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MULTIPLE SCATTERING CORRECTIONS TO THE NUCLEAR OPTICAL POTENTIAL

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MULTIPLE SCATTERING CORRECTIONS TO THE NUCLEAR OPTICAL POTENTIAL

by

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ABSTRACT

It has long proved convenient to describe the elastic scattering of nuclear particles by nuclei in terms of an equivalent two-body potential -- the optical potential. Recent experiments have indicated an apparent discrepancy between the experimentally observed values of the optical potential and the theoretical values predicted by the simple first-order theory. In this paper the leading multiple scattering corrections to the first-order theoretical potential are calculated, and the resulting second-order potential is evaluated for two nuclear models for incident pions and nucleons. It is found that the inclusion of such corrections can bring the theoretical and experimental potentials into agreement.

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I. INTRODUCTION

The optical model description of the scattering of a nuclear particle by a nucleus is concerned primarily with the replacement of the many-body interactions of the incident particle with the nucleons in the target nucleus by the interaction of the incident particle with an equivalent potential -- the optical potential. The term "optical model" is by analogy to the corresponding problem of the propagation of light in a medium, where the many-body interactions of the light with the particles of the medium may, in a certain approximation, be replaced by attributing an index of refraction to the medium as a whole.

For low energy incident particles the problem of determining the equivalent potential has been treated primarily as phenomenological,[1,2] although there has recently been some progress in the relation of the low energy optical potential to the underlying two-particle interactions.[3-5] For high energy incident particles Watson and his collaborators[6-9] have shown that, to a good approximation, the optical potential may be directly related to quantities characterizing the scattering of the incident particles off free nucleons. The approximate evaluation of such an optical potential has been carried through by many authors,[10-14] and the comparison with experiment has been reasonably satisfactory.

An apparent discrepancy has recently been noted, however,[15,16] between these calculated values and the measured values of the optical potential for pions having an energy of a few Bev. This discrepancy seems to indicate that, at 3 Bev, for instance, the imaginary part of the potential is about 20% greater in magnitude than the calculated value. It is shown in this paper that corrections of such magnitude may be obtained by an evaluation of higher order terms in Watson's theory of the optical potential.

It has long been known that such corrections depend on the structure of the scattering medium.[17] Such corrections were estimated by Francis and Watson[8,18] to illustrate the general theory of the optical model potential. More recent studies have been made by Glauber[19] and, in particular, by Bég.[20] These studies do not seem, however, to have emphasized sufficiently the sensitivity of the results to the nuclear model assumed. In this paper we shall discuss this problem more generally and demonstrate explicitly the relations between the various models.

The formal theory of Watson's optical potential will be reviewed in Section II, and an evaluation of the second-order potential is carried through in Section III. Sections IV and V are concerned with the second-order potential for pion-nucleus and nucleon-nucleus scattering, respectively, and Section VI treats corrections to the first-order theory due to the nonlocality in coordinate space of the optical model potential.

We choose throughout units such that $\hbar = c = 1$.

II. FORMAL THEORY OF THE OPTICAL POTENTIAL

We wish to describe the elastic scattering of a particle from a nucleus of mass number A and charge Z. By "elastic," we mean that scattering which does not change the energy state of the nucleus. To accomplish this, we introduce the nuclear Hamiltonian H_N with the complete set of eigenstates $g_\gamma(\xi)$ -- where ξ is some complete set of nuclear coordinates and $g_\gamma(\xi) \equiv \langle \xi | \gamma \rangle$ -- and the corresponding energy eigenvalues W_γ. Then

$$H_N g_\gamma(\xi) = W_\gamma g_\gamma(\xi), \tag{2.1}$$

and we choose W_0 to be the nuclear state of lowest energy -- the ground state. Similarly, the free state of the incident particle is described by a Hamiltonian h possessing the eigenstates $\varphi_q(\vec{x})$ and associated energies ϵ_q. The interaction of the incident particle with the nucleus is assumed to be of the form

$$V = \sum_{\alpha=1}^{A} v_\alpha. \tag{2.2}$$

The Schrödinger equation which describes the scattering of an incident particle in the particle-nucleus barycentric system is then

$$(H_0 + V)\Psi_a = E_a\Psi_a, \tag{2.3}$$

where

$$H_0 = H_N + h, \qquad E_a = W_0 + \epsilon_0. \tag{2.4}$$

Here W_0 and ϵ_0 are, respectively, the energies of the nucleus and particle in the particle-nucleus barycentric system. Equation (2.3) is to be solved subject to the boundary condition that, at large distances of the incident particle from the nucleus,

$$\Psi_a \rightarrow g_0(\xi)\ \varphi_{q_0}(\vec{x}). \tag{2.5}$$

The Møller wave matrix Ω may then be introduced as usual,

$$\Psi_a = \Omega\ g_0(\xi)\ \varphi_{q_0}(\vec{x}), \tag{2.6}$$

and the Schrödinger equation (2.3) for Ψ_a with the boundary conditions (2.5) may be converted into an integral equation for Ω in the usual way.[21]

$$\begin{aligned} \Omega &= 1 + \frac{1}{a} V\Omega, \\ a &= E_a + i\eta - H_0. \end{aligned} \tag{2.7}$$

(Here η is a positive, infinitesimal parameter introduced for performing the integrations across the pole of a^{-1}.)

Watson[6] has shown that equation (2.7) for Ω formally has "multiple-scattering" solutions of the form

$$\Omega = 1 + \frac{1}{a} \sum_{\alpha=1}^{A} t_\alpha \Omega_\alpha,$$

$$\Omega_\alpha = 1 + \frac{1}{a} \sum_{\beta \neq \alpha = 1}^{A} t_\beta \Omega_\beta, \tag{2.8}$$

$$t_\alpha = v_\alpha + v_\alpha \frac{1}{a} t_\alpha .$$

For the study of elastic scattering[8] we are interested in the matrix elements of the scattering operator $T = V\Omega$ between nuclear states of equal energy. Introducing the notation $< \ldots >$ to designate such matrix elements, we define quantities

$$T_c = < T >, \qquad \Omega_c = < \Omega >, \qquad F_c = < \Omega_\alpha >, \tag{2.9}$$

operators on the coordinates of the incident particle. Note that F_c is independent of the index "α," as we are dealing with completely anti-symmetrized nuclear wave functions.

If we can find a "potential" $\mathcal{O}$ such that

$$T_c = \langle V\Omega \rangle = \mathcal{O}\Omega_c = \mathcal{O} + \mathcal{O}\frac{1}{\epsilon_0 + i\eta - h} T_c, \tag{2.10}$$

then equation (2.7) reduces immediately to a two-body equation for the scattering of the incident particle by the potential $\mathcal{O}$:

$$\Omega_c = \langle \Omega \rangle = 1 + \frac{1}{\epsilon_0 + i\eta - h} \langle V\Omega \rangle = 1 + \frac{1}{\epsilon_0 + i\eta - h}\mathcal{O}\Omega_c, \tag{2.11}$$

corresponding to a Schrödinger equation

$$(h + \mathcal{O})\varphi_{q_0} = \epsilon_0\varphi_{q_0}. \tag{2.12}$$

$\mathcal{O}$ is the so-called "optical potential," by analogy to the treatment of the coherent scattering of light by a medium in terms of an index of refraction. Clearly, $\mathcal{O}$ depends on the initial state of the nucleus, here taken to be $g_0(\xi)$, and, in general, depends on the complete solution of the many-body problem.[22] For high energy incident particles, however, $\mathcal{O}$ may be, to a good approximation, related to the scattering of the incident particle off free nucleons and to quantities describing the initial state of the nucleus.

To determine $\mathcal{O}$, we note

$$V\Omega = \sum_{\alpha=1}^{A} v_\alpha\Omega = \sum_{\alpha=1}^{A} v_\alpha\left(1 + \frac{1}{a}\,t_\alpha\right)\Omega_\alpha = \sum_{\alpha=1}^{A} t_\alpha\Omega_\alpha \tag{2.13}$$

by equations (2.2) and (2.8). Introducing the projection operator P_{ND} off of the initial nuclear state,

$$(1 - P_{ND})\, g_\gamma(\xi) = \delta_{\gamma 0} g_\gamma(\xi), \tag{2.14}$$

we may write

$$T_c = < V\Omega > = \sum_{\alpha=1}^{A} < tt_\alpha \Omega_\alpha > = \sum_{\alpha=1}^{A} < tt_\alpha P_{ND} \Omega_\alpha > + \sum_{\alpha=1}^{A} < tt_\alpha (1 - P_{ND}) \Omega_\alpha >. \tag{2.15}$$

Defining G_α by

$$P_{ND}\Omega_\alpha = (G_\alpha - 1)(1 - P_{ND})\Omega_\alpha, \tag{2.16}$$

so

$$G_\alpha = 1 + \frac{1}{a} \sum_{\beta \neq \alpha = 1}^{A} P_{ND} tt_\beta G_\beta, \tag{2.17}$$

and substituting into equation (2.15), we find

$$T_c = \sum_{\alpha=1}^{A} < tt_\alpha G_\alpha (1 - P_{ND}) \Omega_\alpha > = \sum_{\alpha=1}^{A} < tt_\alpha G_\alpha > F_c = \mathcal{V} F_c. \tag{2.18}$$

F_c satisfies the equation

$$F_c = \langle \Omega_\alpha \rangle = 1 + \frac{1}{\epsilon_0 + i\eta - h} \sum_{\beta \neq \alpha = 1}^{A} \langle t_\beta \Omega_\beta \rangle. \tag{2.19}$$

Summing over α and using equation (2.15),

$$AF_c = A + \frac{1}{\epsilon_0 + i\eta - h} (A - 1) T_c,$$

we obtain the equation for T_c,

$$T_c = \mathcal{V} + \mathcal{V} \frac{1}{\epsilon_0 + i\eta - h} (1 - \frac{1}{A}) T_c. \tag{2.20}$$

If A is sufficiently large that we may disregard terms of order A^{-1}, we see, upon comparison of equations (2.20) and (2.10),

$$\mathcal{O} = \mathcal{V} = \sum_{\alpha=1}^{A} \langle t_\alpha G_\alpha \rangle, \tag{2.21}$$

where we use equation (2.18) for $\mathcal{V}$ and G_α is given by equation (2.17).

To keep terms of relative order A^{-1}, define a pseudopotential v and an associated scattering operator T,

$$v = \alpha \mathcal{V}, \qquad T = \alpha T_c, \qquad \alpha = 1 - \frac{1}{A}. \tag{2.22}$$

Then the equation for T is, from equation (2.20),

$$T = \alpha T_c = \alpha \mathcal{V} + \alpha \mathcal{V} \frac{1}{\epsilon_0 + i\eta - h} \alpha T_c = v + v \frac{1}{\epsilon_0 + i\eta - h} T. \tag{2.23}$$

Thus, T is obtained by solving a Schrödinger equation for scattering from the pseudopotential v, given by equations (2.21) and (2.22). The actual scattering amplitude T_c and the differential cross section $\left.\frac{d\sigma}{d\Omega}\right)_c$ (in the particle-nucleus barycentric system) to be compared with experiment are then given by

$$T_c = (1 - \frac{1}{A})^{-1} T,$$

$$\left.\frac{d\sigma}{d\Omega}\right)_c = (1 - \frac{1}{A})^{-2} (2\pi)^4 |T|^2 \left(\frac{\epsilon_0 W_0}{\epsilon_0 + W_0}\right)^2, \qquad (2.24)$$

where we have used equation (A-6) of Appendix A.

III. EVALUATION OF THE SECOND-ORDER OPTICAL POTENTIAL

From equations (2.17) and (2.21) we see that

$$\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2 + \ldots = \sum_{\alpha=1}^{A} \langle 0 | t_\alpha G_\alpha | 0 \rangle$$

$$= \sum_{\alpha=1}^{A} \langle 0 | t_\alpha | 0 \rangle + \sum_{\alpha\neq\beta=1}^{A} \langle 0 | t_\alpha \frac{1}{a} P_{ND} t_\beta | 0 \rangle + \ldots \tag{3.1}$$

and so is expressed as a series of scattering operators -- describing the scattering of the incident particle by the α^{th} bound nucleon -- averaged over the ground state of the nucleus.

For high energy incident particles, we may make the so-called "impulse approximation"[23] in which we replace the bound scattering operators in equation (3.1) by the scattering operators for the scattering of the incident particle by a free nucleon. The relative error incurred from this approximation is[7,24]

$$\delta_0 \approx \frac{B_{av}^2}{\epsilon_0^2} \left(\frac{f}{\lambda\!\!\!^{-}}\right), \tag{3.2}$$

where B_{av} is the average binding potential of a nucleon in the nucleus, f is the scattering amplitude, and $\lambda\!\!\!^{-}$ is the reduced de Broglie wavelength of the incident particle. For high energy incident particles this error is expected to be small.

Thus, if the initial momenta of the nucleon and particle are $\vec{P}_{\alpha 0}$ and $\vec{q}_0$, respectively, and the scattering leads to the final state $\vec{P}_\alpha$ and $\vec{q}$, we may write

$$tt_\alpha = < \vec{P}_\alpha, \vec{q} \mid t_\alpha \mid \vec{P}_{\alpha 0}, \vec{q}_0 > \delta(\vec{P}_\alpha + \vec{q} - \vec{P}_{\alpha 0} - \vec{q}_0), \tag{3.3}$$

where t_α is the free nucleon scattering amplitude, defined only on the "momentum shell." It is frequently convenient to consider tt_α and t_α as operating on the nuclear coordinates but as functions of $\vec{q}$ and $\vec{q}_0$. When we wish to do this we write

$$tt_\alpha = tt_\alpha(\vec{q}, \vec{q}_0), \qquad t_\alpha = t_\alpha(\vec{q}, \vec{q}_0). \tag{3.4}$$

By carrying out the Fourier expansion of the initial nuclear wave function in the momentum of the α^{th} nucleon and using equation (3.3), one obtains[24]

$$tt_\alpha |0> = e^{-i(\vec{q} - \vec{q}_0)\cdot \vec{z}_\alpha} t_\alpha |0> = e^{i\vec{\kappa}\cdot\vec{z}_\alpha} t_\alpha |0>, \tag{3.5}$$

where $\vec{z}_\alpha$ is the coordinate of nucleon "α," and t_α still operates on $|0>$

through its dependence on $\vec{P}_{\alpha 0}$. This dependence on $\vec{P}_{\alpha 0}$ is in terms of relative momenta only, however, so for $\vec{q}_0$ much greater than the average nucleon momentum in the initial nuclear state we may ignore the dependence of t_α on $\vec{P}_{\alpha 0}$. Then t_α operates on $|0>$ only through its spin and isospin coordinates.

The calculation of the first-order optical potential $<\vec{q}|\mathcal{V}_1|\vec{q}_0>$, defined between momentum states $\vec{q}$ and $\vec{q}_0$ of the incident particle, now follows directly:

$$<\vec{q}|\mathcal{V}_1|\vec{q}_0> = \sum_{\alpha=1}^{A} <0|\mathrm{t}_\alpha(\vec{q}, \vec{q}_0)|0> = \sum_{\alpha=1}^{A} <0|e^{i\vec{\kappa}\cdot\vec{z}_\alpha} t_\alpha(\vec{q}, \vec{q}_0)|0>. \tag{3.6}$$

Introducing the nucleon density distributions

$$P(\vec{x}) = <0|\delta(\vec{x} - \vec{z}_\alpha)|0>, \qquad \int d^3x\, P(\vec{x}) = 1, \tag{3.7a}$$

$$\rho(\vec{x}) = V_A\, P(\vec{x}), \qquad \int d^3x\, \rho(\vec{x}) = V_A = \frac{4\pi}{3} R_A^3, \tag{3.7b}$$

and employing the notation $(0|t_\alpha|0)$ to designate matrix elements of t_α between the nuclear ground state spin and isospin wave functions, we obtain

$$<\vec{q}|\mathcal{V}_1|\vec{q}_0> = (0|t(\vec{q}, \vec{q}_0)|0)\frac{A}{V_A} \int d^3z\, \rho(\vec{z})\, e^{i\vec{\kappa}\cdot\vec{z}}$$

$$= (0|t(\vec{q}, \vec{q}_0)|0)\, A\, C(|\vec{\kappa}|), \tag{3.8}$$

where

$$C(|\vec{\kappa}|) = \int d^3z \; P(\vec{z}) \; e^{i\vec{\kappa}\cdot\vec{z}}. \tag{3.9}$$

Since $\rho(\vec{z})$ is sensibly different from zero only for $|\vec{z}| \lesssim R_A$, $C(|\vec{\kappa}|)$ will be small unless

$$|\vec{\kappa}| = |\vec{q}_0 - \vec{q}| = 2q_0 \sin\frac{\theta}{2} \lesssim \frac{1}{R_A}, \qquad \theta \lesssim \frac{1}{q_0 R_A}, \tag{3.10}$$

where θ is the angle of scattering in the barycentric system. Thus, for q_0 large, θ may be small enough that

$$t(\vec{q}, \vec{q}_0) \cong t(\vec{q}_0, \vec{q}_0) \equiv t^0(q_0). \tag{3.11}$$

If we assume this, we may obtain the coordinate representative of $\mathcal{V}_1$ as follows:

$$\langle \vec{x}|\mathcal{V}_1|\vec{q}_0\rangle = \int \langle \vec{x}|\vec{q}\rangle \, d^3q \, \langle \vec{q}|\mathcal{V}_1|\vec{q}_0\rangle$$

$$= (0|t^0(q_0)|0) \left(\frac{A}{V_A}\right) \int \frac{e^{i\vec{q}\cdot\vec{x}}}{(2\pi)^{3/2}} \, d^3q \int e^{-i(\vec{q}-\vec{q}_0)\cdot\vec{z}} \rho(\vec{z}) d^3z$$

$$= (2\pi)^3 \, (0|t^0(q_0)|0) \left(\frac{A}{V_A}\right) \rho(\vec{x}) \, \frac{e^{i\vec{q}_0\cdot\vec{x}}}{(2\pi)^{3/2}} \, . \tag{3.12}$$

But

$$< \vec{x}|\mathcal{V}_1|\vec{q}_0 > = \int < \vec{x}|\mathcal{V}_1|\vec{x}' > d^3x' < \vec{x}'|\vec{q}_0 > = \int < \vec{x}|\mathcal{V}_1|\vec{x}' > d^3x' \frac{e^{i\vec{q}_0 \cdot \vec{x}'}}{(2\pi)^{3/2}} . \tag{3.13}$$

Comparing equations (3.12) and (3.13), we find

$$\begin{aligned} < \vec{x}|\mathcal{V}_1|\vec{x}' > &= (2\pi)^3 \, (0|t^0(q_0)|0) \, (\frac{A}{V_A}) \, \rho(\vec{x}) \, \delta(\vec{x} - \vec{x}') \\ &\equiv \mathcal{V}_1(\vec{x}) \, \delta(\vec{x} - \vec{x}'). \end{aligned} \tag{3.14}$$

From equation (A-9) of Appendix A:

$$t^0(q_0) = \frac{-1}{(2\pi)^2 \, \epsilon_0} \, f_L^0(q_0), \tag{3.15}$$

where $f_L^0(q_0)$ is the laboratory forward-scattering amplitude for the incident particle on a free nucleon. Then

$$\begin{aligned} \mathcal{V}_1(\vec{x}) &= V_1 \, \rho(\vec{x}), \\ V_1 &= - \frac{2\pi}{\epsilon_0} \, (\frac{A}{V_A}) \, f_L^0(q_0), \end{aligned} \tag{3.16}$$

V_1 depending parametrically on q_0, the momentum of the incident particle.

To evaluate the second-order potential, we use equation (2.14) for P_{ND} in equation (3.1) to obtain

$$< \vec{q}|\mathcal{V}_2|\vec{q}_0 > = \sum_{\alpha\neq\beta=1}^{A} \sum_{\gamma\neq 0} \int \frac{d^3q'}{\epsilon_0 + W_0 + i\eta - \epsilon_{q'} - W_\gamma}$$

$$\times < 0|\mathrm{tt}_\alpha(\vec{q},\vec{q}\,')|\gamma > < \gamma|\mathrm{tt}_\beta(\vec{q}\,',\vec{q}_0)|0 >. \qquad (3.17)$$

We first simplify equation (3.17) by neglecting the energy difference $(W_0 - W_\gamma)$ in the denominator. For sufficiently high energies this is justified, since the resulting error is of the order[24,25]

$$\delta_1 \approx \frac{1}{3} \frac{K_{av}}{\epsilon_0}, \qquad (3.18)$$

where $K_{av} \approx 30$ Mev is the average kinetic energy of a nucleon bound in the nucleus. Employing closure to perform the sum over γ in equation (3.17) and using equation (3.5), we find

$$< \vec{q}|\mathcal{V}_2|\vec{q}_0 > = \sum_{\alpha\neq\beta=1}^{A} \int \frac{d^3q'}{\epsilon_0 + i\eta - \epsilon_{q'}}$$

$$\times \Big[< 0|e^{-i(\vec{K}'\cdot\vec{z}_\alpha + \vec{K}\cdot\vec{z}_\beta)} t_\alpha(\vec{q},\vec{q}\,')\ t_\beta(\vec{q}\,',\vec{q}_0)|0 >$$

$$- < 0|e^{-i\vec{K}'\cdot\vec{z}_\alpha} t_\alpha(\vec{q},\vec{q}\,')|0 > < 0|e^{-i\vec{K}\cdot\vec{z}_\beta} t_\beta(\vec{q}\,',\vec{q}_0)|0 > \Big]$$

(3.19)

where

$$\vec{K}' = \vec{q} - \vec{q}', \qquad \vec{K} = \vec{q}' - \vec{q}_0. \tag{3.20}$$

We now utilize a technique due to Lax and Feshbach[24,26] for the evaluation of matrix elements such as those occurring in equation (3.19). The quantity $t_\alpha t_\beta$ is a matrix in spin and isospin space, but is no longer an operator on the nuclear coordinates -- by our assumptions following equation (3.5). For each of the four possible states of the nucleon pair (α,β), we may expect a distinct pair-correlation function. The evaluation of the desired matrix elements involves giving the appropriate weight to each such state of a pair (α,β). For simplicity, we shall follow Lax and Feshbach[24,26] and assume only two correlation functions: one for space-symmetric and one for space-antisymmetric states.

Let $P_{\alpha\beta}$ be the space-exchange operator which interchanges $\vec{z}_\alpha$ and $\vec{z}_\beta$, and write

$$|s\rangle = \frac{1}{2}(1 + P_{\alpha\beta})|0\rangle,$$

$$|a\rangle = \frac{1}{2}(1 - P_{\alpha\beta})|0\rangle. \tag{3.21}$$

Then, if $\mathcal{O}_{\alpha\beta}$ is an operator symmetric in (α,β) we have

$$\langle 0|\mathcal{O}_{\alpha\beta}|0\rangle = \langle s|\mathcal{O}_{\alpha\beta}|s\rangle + \langle a|\mathcal{O}_{\alpha\beta}|a\rangle. \tag{3.22}$$

We introduce the corresponding pair distribution functions

$$P_{2s}(\vec{x}, \vec{x}') = \frac{1}{2} < 0|(1 + P_{\alpha\beta})\, \delta(\vec{x} - \vec{z}_\alpha)\, \delta(\vec{x}' - \vec{z}_\beta)|0 >,$$

$$P_{2a}(\vec{x}, \vec{x}') = \frac{1}{2} < 0|(1 - P_{\alpha\beta})\, \delta(\vec{x} - \vec{z}_\alpha)\, \delta(\vec{x}' - \vec{z}_\beta)|0 >, \qquad (3.23)$$

which may evidently be written in the general form

$$P_{2s}(\vec{z}_\alpha, \vec{z}_\beta) = P(\vec{z}_\alpha) \left[P(\vec{z}_\beta) + Q_s(\vec{z}_\alpha - \vec{z}_\beta, \vec{z}_\beta) \right],$$

$$P_{2a}(\vec{z}_\alpha, \vec{z}_\beta) = P(\vec{z}_\alpha) \left[P(\vec{z}_\beta) + Q_a(\vec{z}_\alpha - \vec{z}_\beta, \vec{z}_\beta) \right], \qquad (3.24)$$

where $P(\vec{z})$ is defined by equation (3.7), and the functions Q_s and Q_a represent conditional probability distributions. For a sufficiently large nucleus we would anticipate being able to set

$$Q_s(\vec{z}_\alpha - \vec{z}_\beta, \vec{z}_\beta) = P(\vec{z}_\beta)\, G_s(\vec{z}_\alpha - \vec{z}_\beta),$$

$$Q_a(\vec{z}_\alpha - \vec{z}_\beta, \vec{z}_\beta) = P(\vec{z}_\beta)\, G_a(\vec{z}_\alpha - \vec{z}_\beta). \qquad (3.25)$$

We now identify the operator $\mathcal{O}_{\alpha\beta}$ of equation (3.22) with the expression

$$e^{-i(\vec{K}'\cdot\vec{z}_\alpha + \vec{K}\cdot\vec{z}_\beta)}\; t_\alpha t_\beta$$

in equation (3.19). In evaluating equation (3.19) we must then consider terms like

$$< s|e^{-i(\vec{K}'\cdot\vec{z}_\alpha+\vec{K}\cdot\vec{z}_\beta)}\, t_\alpha t_\beta|s >$$

$$= (s|t_\alpha t_\beta|s)\int P_{2s}(\vec{z}_\alpha,\vec{z}_\beta)e^{-i(\vec{K}'\cdot\vec{z}_\alpha+\vec{K}\cdot\vec{z}_\beta)}\, d^3z_\alpha d^3z_\beta. \qquad (3.26)$$

Now

$$\int\frac{d^3q'}{\epsilon_0+i\eta-\epsilon_{q'}}\int d^3z_\alpha e^{-i\vec{K}'\cdot\vec{z}_\alpha}\, P(\vec{z}_\alpha)\int d^3z_\beta e^{-i\vec{K}\cdot\vec{z}_\beta}\, P(\vec{z}_\beta)(s|t_\alpha t_\beta|s)$$

$$\approx B(s|t_\alpha^0 t_\beta^0|s), \qquad (3.27)$$

where

$$B \equiv \int\frac{d^3q'}{\epsilon_0+i\eta-\epsilon_{q'}}\; C(|\vec{K}'|)\; C(|\vec{K}|). \qquad (3.28)$$

Here $C(|\vec{K}|)$ is defined by equation (3.9), and we have made use of the approximation of equation (3.11) to set $t_\alpha = t_\alpha^0$ and $t_\beta = t_\beta^0$.

It is convenient to introduce

$$J_s = \sum_{\alpha\neq\beta=1}^{A}\int\frac{d^3q'}{\epsilon_0+i\eta-\epsilon_{q'}}\, d^3z_\alpha d^3z_\beta\; P(\vec{z}_\alpha)\; Q_s(\vec{z}_\alpha-\vec{z}_\beta,\vec{z}_\beta)$$

$$\times\, e^{-i(\vec{K}'\cdot\vec{z}_\alpha+\vec{K}\cdot\vec{z}_\beta)}\;(s|t_\alpha(\vec{q},\vec{q}')\; t_\beta(\vec{q}',\vec{q}_0)|s) \qquad (3.29)$$

and a similar quantity J_a for the space-antisymmetric states. Then, if we define a quantity Δ,

$$\Delta \equiv \sum_{\alpha\neq\beta=1}^{A} \left[(0|t_\alpha^0 t_\beta^0|0) - (0|t_\alpha^0|0)(0|t_\beta^0|0) \right] , \tag{3.30}$$

equation (3.19) may be rewritten as

$$< \vec{q}|\mathcal{V}_2|\vec{q}_0 > = B\Delta + J_s + J_a . \tag{3.31}$$

To simplify J_s and J_a, let

$$\vec{r} = \vec{z}_\alpha - \vec{z}_\beta , \qquad \vec{z} = \vec{z}_\alpha .$$

Then

$$J_s \cong \int d^3z\, P(\vec{z}) e^{-i(\vec{q}-\vec{q}_0)\cdot\vec{z}} \int \frac{d^3q'}{\epsilon_0 + i\eta - \epsilon_{q'}} e^{i\vec{q}'\cdot\vec{r}}$$

$$\times \left[\sum_{\alpha\neq\beta=1}^{A} (s|t_\alpha(\vec{q}_0,\vec{q}')t_\beta(\vec{q}',\vec{q}_0)|s) \right] d^3r\, e^{-i\vec{q}_0\cdot\vec{r}} Q_s(\vec{r},\vec{z}), \tag{3.32}$$

where we have replaced $\vec{q}$ by $\vec{q}_0$ in the scattering operators because of the first factor in the equation and the arguments leading to equation (3.11). For incident particles of energy high enough that $q_0R_s \gg 1$ and $q_0R_a \gg 1$, where R_s and R_a are defined by equation (3.34) below, J_s and J_a may be approximately evaluated to give

$$J_s \cong \int d^3z\, P(\vec{z})\, e^{-i(\vec{q}-\vec{q}_0)\cdot\vec{z}}\, \frac{(2\pi)^3\epsilon_0}{iq_0V_A}\, R_s(\vec{z}) \sum_{\alpha\neq\beta=1}^{A} (s|t^0_\alpha t^0_\beta|s),$$

$$J_a \cong \int d^3z\, P(\vec{z})\, e^{-i(\vec{q}-\vec{q}_0)\cdot\vec{z}}\, \frac{(2\pi)^3\epsilon_0}{iq_0V_A}\, R_a(\vec{z}) \sum_{\alpha\neq\beta=1}^{A} (a|t^0_\alpha t^0_\beta|a), \tag{3.33}$$

where we neglect terms of relative order $(q_0R_s)^{-2}$. Here we have introduced "correlation lengths" R_s and R_a, defined by

$$\frac{1}{V_A} R_s(\vec{z}) = \int_0^\infty dr\, Q_s(r\hat{q}_0,\vec{z}),$$

$$\frac{1}{V_A} R_a(\vec{z}) = \int_0^\infty dr\, Q_a(r\hat{q}_0,\vec{z}). \tag{3.34}$$

Except for small nuclei, equation (3.25) should be a good approximation, and we may write

$$R_s = \int_0^\infty G_s(r)dr,$$

$$R_a = \int_0^\infty G_a(r)dr, \tag{3.35}$$

independent of $\vec{z}$.

Writing

$$\sum_{\alpha\neq\beta=1}^{A} (0|t^0_\alpha t^0_\beta|0) = \frac{A^2}{(2\pi)^4\,\epsilon_0^2}\, S,$$

$$\sum_{\alpha\neq\beta=1}^{A} (0|t_\alpha^0 t_\beta^0 \, P_{\alpha\beta}|0) = \frac{A^2}{(2\pi)^4 \, \epsilon_0^2} S_\tau, \tag{3.36}$$

we obtain

$$\sum_{\alpha\neq\beta=1}^{A} (s|t_\alpha^0 t_\beta^0|s) = \frac{A^2}{(2\pi)^4 \, \epsilon_0^2} \; \frac{1}{2} (S + S_\tau),$$

$$\sum_{\alpha\neq\beta=1}^{A} (a|t_\alpha^0 t_\beta^0|a) = \frac{A^2}{(2\pi)^4 \, \epsilon_0^2} \; \frac{1}{2} (S - S_\tau). \tag{3.37}$$

Finally, using equations (3.37), (3.35), and (3.33) in equation (3.31) the result is obtained

$$< \vec{q}|\mathcal{V}_2|\vec{q}_0 > = B\Delta + \left[\frac{V_A}{(2\pi)^3}\right] \; C(|\vec{q} - \vec{q}_0|)V_2, \tag{3.38}$$

where

$$V_2 = \frac{(2\pi)^2}{2i\epsilon_0 q_0} \left(\frac{A}{V_A}\right)^2 \left[(R_s + R_a)S + (R_s - R_a)S_\tau\right]. \tag{3.39}$$

Thus, equation (3.1) becomes (valid to our approximation)

$$< \vec{q}|\mathcal{V}|\vec{q}_0 > = C(|\vec{q} - \vec{q}_0|) \left[\frac{V_A}{(2\pi)^3}\right] \; (V_1 + V_2) + B\Delta. \tag{3.40}$$

The term involving B is cumbersome to use. Since it is of order A^{-1} compared to V_2, it may be discarded for large nuclei. It may also be transformed into a more convenient form. To see this, let us introduce equation (3.40) into equation (2.20) and iterate once.

$$< \vec{q}|T_c|\vec{q}_0 > = C(|\vec{q}-\vec{q}_0|) \left[\frac{V_A}{(2\pi)^3}\right] (V_1 + V_2) + B\left\{\Delta + (1 - \frac{1}{A}) \left[\frac{V_A}{(2\pi)^3} V_1\right]^2\right\} + \cdots \quad . \tag{3.41}$$

Now we attempt to obtain the same result by defining a new potential and scattering operator:

$$< \vec{q}|v|\vec{q}_0 > = C(|\vec{q}-\vec{q}_0|) \left[\frac{V_A}{(2\pi)^3}\right] \alpha(V_1 + V_2), \tag{3.42a}$$

$$T = v + v \frac{1}{\epsilon_0 + i\eta - h} T, \tag{3.42b}$$

$$T_c = \beta T, \tag{3.42c}$$

where α and β are to be determined. In order that equations (3.42c) and (3.41) be consistent, we must have

$$\beta = \alpha^{-1}.$$

To first-order in A^{-1} and in the potential strengths we find that

$$\alpha = 1 - \frac{1}{A} + \frac{\Delta}{\left[\frac{V_A}{(2\pi)^3} V_1\right]^2} \quad . \tag{3.43}$$

To obtain the differential cross section $\left.\frac{d\sigma}{d\Omega}\right)_c$ to be compared with experiment, we proceed in a fashion analogous to the discussion following equation (2.23). We use the potential v defined by equation (3.42a) in the Schrodinger equation (2.12) to obtain a differential cross section $\frac{d\sigma}{d\Omega}$. Then

$$\left.\frac{d\sigma}{d\Omega}\right)_c = |\alpha|^{-2} \frac{d\sigma}{d\Omega} . \tag{3.44}$$

Proceeding as we did in obtaining equation (3.16) from (3.8), the coordinate representative of v is

$$< \vec{x}|v|\vec{x}' > = v(\vec{x})\ \delta(\vec{x} - \vec{x}'), \tag{3.45}$$

where

$$v(\vec{x}) = v\ \rho(\vec{x}) = \alpha(V_1 + V_2)\ \rho(\vec{x}). \tag{3.46}$$

$v(\vec{x})$ is now our general result for the second-order optical potential, where we have neglected corrections of relative order A^{-2} and further terms of order

$$\delta_2 \simeq \frac{1}{\epsilon_0 q_0^2} \left(\frac{A}{V_A}\right)^3 (R_s \pm R_a)^2 f^3, \tag{3.47}$$

and f is a typical particle-nucleon scattering amplitude.

IV. THE PION-NUCLEUS OPTICAL POTENTIAL

Expressions for the pion-nucleus first-order optical potential in terms of pion-nucleon scattering amplitudes have been given by several authors.[10,11] These are of particular interest since the relevant scattering amplitudes may be obtained directly from measured pion-nucleon scattering cross sections with the use of dispersion relations.[27,28]

To evaluate the general expressions (3.16), (3.39), and (3.46), we must consider in some detail the pion-nucleon scattering operators t^0. It is convenient to project t^0 onto the isospin substates corresponding to $I = 3/2$ and $I = 1/2$. This may be effected with the respective projection operators $\Lambda_{3/2}$ and $\Lambda_{1/2}$, so

$$t^0 = t^0(\tfrac{3}{2})\, \Lambda_{3/2} + t^0(\tfrac{1}{2})\, \Lambda_{1/2}. \tag{4.1}$$

The laboratory system scattering amplitudes for the $I = 3/2$ and the $I = 1/2$ states are, respectively [from equation (A-9) of Appendix A]:

$$f^0_{3/2} = -(2\pi)^2\, \epsilon_0 t^0(\tfrac{3}{2}),$$

$$f^0_{1/2} = -(2\pi)^2\, \epsilon_0 t^0(\tfrac{1}{2}). \tag{4.2}$$

In terms of these, we define

$$f \equiv \frac{1}{3}\,(2f^0_{3/2} + f^0_{1/2}),$$

$$f_\tau \equiv \frac{1}{3}\,(f^0_{3/2} - f^0_{1/2}). \qquad (4.3)$$

Then, from Appendix B,

$$(0|f^0_L|0) = f \pm \frac{2T_3}{A}\, f_\tau, \qquad (4.4)$$

where ($\pm$) refers to π^+ or π^- mesons, respectively, and T_3 is the third component of the nuclear isotopic spin; that is

$$T_3 = \frac{1}{2}\,(Z - N). \qquad (4.5)$$

Here N is the number of neutrons in the nucleus.

In addition to being scattered, a meson may be absorbed by a nucleus. It is understood that the effect of this is also included in the optical potential. The effective cross section for absorption, per nucleon, is conventionally[29] written as $\Gamma(\sigma_d/2)$, where σ_d is the cross section for absorption on a deuteron and $\Gamma \approx 4$ (reference 30). The mean free path for absorption of a meson in nuclear matter is then λ_a, where[11]

$$\frac{1}{\lambda_a} = \frac{A}{V_A}\,\Gamma\,\frac{\sigma_d}{2}\,. \qquad (4.6)$$

[In adopting this expression we are assuming $T_3 = 0$. This is justified since the contribution from equation (4.6) to V_1 is both small and poorly known for pions in the Bev range.] Following the analysis of Cronin et al.[15] and of Beg,[20] we shall take

$$\sigma_d \approx 0.5 \times 10^{-27}\ \mathrm{cm}^2 \tag{4.7}$$

for pions in the 1 to 5 Bev range.

The expression (3.16) for V_1 then becomes

$$V_1(\pi^{\pm}, q_0) = -\frac{2\pi}{\epsilon_0}\left(\frac{A}{V_A}\right)\left(f \pm \frac{2T_3}{A} f_\tau + i\,\frac{q_0 \Gamma\, \sigma_d}{8\pi}\right), \tag{4.8}$$

where $V_1(\pi^{\pm}, q_0)$ refers to π^+ or π^- mesons, respectively.

In Appendix B, the evaluation of the quantities S, S_τ, and Δ for pion-nucleus scattering is presented. In terms of these results, our expression (3.39) for V_2 becomes

$$V_2(\pi^{\pm},q_0) = \frac{i(2\pi)^2}{2\epsilon_0\, q_0}\left(\frac{A}{V_A}\right)^2 \left\{-\,(R_s + R_a)\left[f^2\left(1 - \frac{1}{A}\right) - \frac{2}{A} f_\tau^2 \pm \frac{4T_3}{A} f\, f_\tau\right]\right.$$
$$\left. +\,(R_s - R_a)\left[\frac{1}{4} f^2\left(1 - \frac{16}{A}\right) + \frac{1}{2} f_\tau^2 \pm \frac{2T_3}{A} f\, f_\tau\right]\right\}, \tag{4.9}$$

and v of equation (3.46) is given as

$$v(\pi^{\pm},q_0) = \left[1 - \frac{1}{A}\left(1 + 2\,\frac{f_\tau^2}{f^2}\right)\right]\left[V_1(\pi^{\pm},q_0) + V_2(\pi^{\pm},q_0)\right]. \tag{4.10}$$

The real parts of the amplitudes f and f_τ may be evaluated using the dispersion relations.[27,28] The imaginary parts are given by the optical theorem in terms of pion-nucleon scattering cross sections. We find, then,

$$f = \frac{1}{2}(D^+ + D^-) + \frac{iq_0}{8\pi}(\sigma^+ + \sigma^-),$$

$$f_\tau = \frac{1}{2}(D^+ - D^-) + \frac{iq_0}{8\pi}(\sigma^+ - \sigma^-). \qquad (4.11)$$

The amplitudes D^+ and D^- have been defined in references (27) and (28). They are the real parts, respectively, of the forward scattering amplitude of $\pi^\pm$ mesons on protons in the laboratory system. σ^+ and σ^- are the corresponding total cross sections for π^+ or π^- proton scattering.[31] The quantities D^+ and D^- have been evaluated only up to 2.6 Bev.[28] Since they are small at these high energies, we have extrapolated them as constant in the range 2.6 to 5 Bev.

Equation (4.10) may now be evaluated. We write it in the form (to order A^{-1}):

$$V(\pi^\pm, q_0) = \frac{1}{\lambda^3}\left[\left(V^R_{00} + \frac{1}{A}V^R_{01} \pm \frac{T_3}{A}V^R_{02}\right) + i\left(V^I_{00} + \frac{1}{A}V^I_{01} \pm \frac{T_3}{A}V^I_{02}\right)\right]$$

$$+ \frac{1}{\lambda^6}\,\frac{(R_s + R_a)}{10^{-13}}\left[\left(V^R_{10} + \frac{1}{A}V^R_{11} \pm \frac{T_3}{A}V^R_{12}\right) + i\left(V^I_{10} + \frac{1}{A}V^I_{11} \pm \frac{T_3}{A}V^I_{12}\right)\right]$$

$$+ \frac{1}{\lambda^6}\,\frac{(R_s - R_a)}{10^{-13}}\left[\left(V^R_{20} + \frac{1}{A}V^R_{21} \pm \frac{T_3}{A}V^R_{22}\right) + i\left(V^I_{20} + \frac{1}{A}V^I_{21} \pm \frac{T_3}{A}V^I_{22}\right)\right].$$

(4.12)

Here λ is related to the nuclear radius R_A [equation (3.7b)] by

$$R_A = 1.2\lambda A^{1/3} \times 10^{-13} \text{ cm.} \tag{4.13}$$

The quantities appearing in equation (4.12) have been evaluated for pion kinetic energies in the range 1 to 5 Bev and are listed in Table I.

For a Fermi gas model of the nucleus, we determine the quantities R_s and R_a. Evaluating equations (3.23) in terms of the single-particle states

$$\varphi_\ell(1) = \frac{1}{V_A^{1/2}} e^{i\vec{k}_\ell \cdot \vec{z}_1} i_\ell(1)\; s_\ell(1), \tag{4.14}$$

where i_ℓ and s_ℓ are the isospin and spin eigenfunctions, respectively, we have

$$P_{2s(a)} = \frac{1}{2} \sum_{\ell \neq m} \Big[< \varphi_\ell(1)\varphi_m(2) | (1 \pm P_{\ell m})\, \delta(\vec{x} - \vec{z}_1)\, \delta(\vec{x}' - \vec{z}_2) | \varphi_\ell(1)\varphi_m(2) >$$

$$- < \varphi_\ell(1)\varphi_m(2) | (1 \pm P_{\ell m})\, \delta(\vec{x} - \vec{z}_1)\, \delta(\vec{x}' - \vec{z}_2) | \varphi_\ell(2)\varphi_m(1) > \Big] . \tag{4.15}$$

Defining

$$\overline{G}_s = \left[\frac{A}{4}\left(\frac{A}{4} - 1\right)\right]^{-1} \sum_{\vec{k}_\ell \neq \vec{k}_m} e^{-i(\vec{k}_\ell - \vec{k}_m)\cdot(\vec{x} - \vec{x}')}, \tag{4.16}$$

TABLE I. The parameters of the $\pi^{\pm}$ nucleus optical model potential as defined by equation (4.12). T_π is the pion kinetic energy in the barycentric system ($T_\pi = \epsilon_0 - \mu$), and R_s and R_a are measured in centimeters.

T_π(Bev)	V^R_{00}(Mev)	V^R_{01}(Mev)	V^R_{02}(Mev)	V^I_{00}(Mev)	V^I_{01}(Mev)	V^I_{02}(Mev)
1.0	11.9	-22.0	-25.5	-47.0	44.7	17.7
1.1	7.8	-12.0	-23.7	-45.5	39.0	6.2
1.2	6.2	-5.0	-22.5	-50.0	43.4	-4.0
1.3	9.4	-7.8	-15.0	-50.5	47.0	-6.7
1.4	13.5	-12.8	-6.4	-49.0	47.4	-5.6
1.5	16.2	-16.0	-3.5	-46.5	44.8	-3.1
1.6	15.6	-16.1	-6.0	-42.9	41.3	+2.4
1.7	12.6	-13.1	-6.1	-41.7	40.1	3.1
1.8	10.6	-11.1	-5.6	-41.0	39.5	3.2
1.9	9.5	-10.0	-5.6	-40.5	39.0	3.3
2.0	8.6	-9.1	-5.9	-40.3	38.7	2.7
2.2	7.4	-7.7	-5.6	-40.5	38.8	1.6
2.4	6.7	-6.8	-5.6	-40.8	39.1	0.4
2.6	6.1	-6.1	-5.4	-41.0	39.2	0.0
3.0	5.3	-5.3	-4.7	-41.0	39.4	0.0
3.5	4.6	-4.5	-4.1	-41.0	39.5	0.0
4.0	4.0	-4.0	-3.6	-41.1	39.6	0.0
4.5	3.6	-3.6	-3.2	-41.1	39.6	0.0
5.0	3.2	-3.2	-2.9	-41.1	39.7	0.0

TABLE I (cont.)

T_π(Bev)	V^R_{10}(Mev)f^{-1}	V^R_{11}(Mev)f^{-1}	V^R_{12}(Mev)f^{-1}	V^I_{10}(Mev)f^{-1}	V^I_{11}(Mev)f^{-1}	V^I_{12}(Mev)f^{-1}
1.0	-2.9	4.0	7.2	5.1	-4.7	-2.7
1.1	-1.8	2.2	5.7	4.9	-4.3	-0.5
1.2	-1.6	1.3	5.5	6.0	-5.4	+1.7
1.3	-2.4	2.1	3.5	6.0	-5.8	2.4
1.4	-3.3	3.2	1.2	5.4	-5.4	1.8
1.5	-3.8	3.7	0.6	4.6	-4.6	1.0
1.6	-3.3	3.4	1.5	3.8	-3.8	-0.0
1.7	-2.6	2.7	1.5	3.6	-3.7	-0.3
1.8	-2.2	2.2	1.3	3.7	-3.7	-0.4
1.9	-1.9	1.9	1.3	3.7	-3.7	-0.9
2.0	-1.7	1.8	1.3	3.7	-3.7	-0.3
2.2	-1.5	1.5	1.2	3.8	-3.7	-0.1
2.4	-1.3	1.4	1.1	3.9	-3.8	+0.1
2.6	-1.2	1.2	1.1	3.9	-3.9	+0.2
3.0	-1.1	1.1	1.0	3.9	-3.9	0.1
3.5	-0.9	0.9	0.8	4.0	-3.9	0.1
4.0	-0.8	0.8	0.7	4.0	-4.0	0.1
4.5	-0.7	0.7	0.7	4.0	-4.0	0.1
5.0	-0.7	0.7	0.6	4.0	-4.0	0.1

TABLE I (cont.)

T_π(Bev)	V_{20}^R(Mev)f^{-1}	V_{21}^R(Mev)f^{-1}	V_{22}^R(Mev)f^{-1}	V_{20}^I(Mev)f^{-1}	V_{21}^I(Mev)f^{-1}	V_{22}^I(Mev)f^{-1}
1.0	1.0	-11.4	-3.6	-1.2	20.4	1.3
1.1	0.6	-7.2	-2.9	-1.1	19.7	0.2
1.2	0.3	-6.2	-2.8	-1.4	24.1	-0.9
1.3	0.5	-9.6	-1.8	-1.5	24.1	-1.2
1.4	0.8	-13.3	-0.6	-1.4	21.6	-0.9
1.5	0.9	-15.0	-0.3	-1.1	18.3	-0.5
1.6	0.9	-13.4	-0.7	-0.9	15.2	+0.0
1.7	0.7	-10.4	-0.7	-0.9	10.4	0.1
1.8	0.6	-8.6	-0.7	-0.9	14.9	0.2
1.9	0.5	-7.6	-0.7	-0.9	14.8	0.5
2.0	0.4	-6.8	-0.6	-0.9	14.7	0.1
2.2	0.4	-6.0	-0.6	-0.9	15.0	0.0
2.4	0.3	-5.4	-0.6	-1.0	15.4	0.0
2.6	0.3	-4.9	-0.6	-1.0	15.6	0.0
3.0	0.3	-4.2	-0.5	-1.0	15.7	0.0
3.5	0.2	-3.7	-0.4	-1.0	15.8	0.0
4.0	0.2	-3.2	-0.4	-1.0	15.9	0.0
4.5	0.2	-2.9	-0.3	-1.0	15.9	0.0
5.0	0.2	-2.6	-0.3	-1.0	15.9	0.0

we obtain

$$P_{2s} = \frac{1}{V_A^2}\left[1 + \frac{1 - (4/A)}{1 + (4/A)}\right]\bar{G}_s, \qquad P_{2a} = \frac{1}{V_A^2}(1 - \bar{G}_s), \tag{4.17}$$

$$\bar{G}_s(r) \cong \frac{9\pi}{2}\,\frac{[J_{3/2}(k_F r)]^2}{(k_F r)^3}, \tag{4.18}$$

where $r = |\vec{x} - \vec{x}'|$ and k_F is the Fermi momentum. Then[32]

$$R_s + R_a = \frac{-(8/A)}{1 + (4/A)}\,\frac{3\pi}{5k_F},$$

$$R_s - R_a = \frac{2}{1 + (4/A)}\,\frac{3\pi}{5k_F}, \tag{4.19}$$

so equation (4.9) becomes

$$V_2^F(\pi^{\pm}, q_0) = i\,\frac{2\pi}{\epsilon_0 q_0}\,\frac{k_F^2}{5}\left(\frac{A}{V_A}\right)\left(f^2 + 2f_\tau^2 \pm \frac{8T_3}{A} f f_\tau\right)[1 - (4/A)]\,\lambda^{-5} \tag{4.20}$$

agreeing with the result of Bég.[20]

It is interesting to note the effect of the Pauli principle on the optical potential. Since the Pauli principle prohibits certain final states of the target nucleon in the nucleus which would be accessible to a free target nucleon, Goldberger[33] has shown that, at lower energies, the net effect of such exclusion is to reduce the effective pion-nucleon cross section from its value for free pion-nucleon scattering, and hence to reduce the magnitude of the imaginary part of the optical potential.

At the higher energies being considered here, however, we see that the effect of the Pauli principle may be in the opposite direction. Thus, from equations (4.8) and (4.20) (ignoring the A^{-1} corrections and assuming $f_T = 0$),

$$\text{Im } V^F = -\frac{2\pi}{\epsilon_0}\left(\frac{A}{V_A}\right)\left[f_I - \frac{k_F^2}{5q_0}(f_R^2 - f_I^2)\right], \qquad (4.21)$$

where $f = f_R + if_I$. From the optical theorem this may be written[19]

$$\text{Im } V^F = -\frac{2\pi}{\epsilon_0}\left(\frac{A}{V_A}\right)\frac{q_0}{4\pi}\left\{\sigma + \frac{k_F^2}{20\pi}\left[\sigma^2 - \left(\frac{4\pi}{q_0} f_R\right)^2\right]\right\}, \qquad (4.22)$$

where σ is the free pion-nucleon total cross section. Thus, if $f_R < f_I$, as is the case at high energies, the net effect of the Pauli principle is to <u>increase</u> the effective pion-nucleon cross section from its value for free pion-nucleon scattering and hence to increase the imaginary part of the optical potential.

The result of Watson[18] follows directly from the nuclear model in which $R_s = R_a$. This assumes that the correlations are similar in spatially symmetric and antisymmetric states.

Brueckner and Gammel[34] describe a wave function of the relative coordinate of a pair of nucleons in nuclear matter. Using their results, we obtain

$$R_s \approx R_a = -0.84 \times 10^{-13} \text{ cm} \qquad (4.23)$$

describing the correlations due to the "hard cores" of the nucleon-nucleon interaction. Some indication of a similar result has recently been deduced from experiment.[24]

The pion optical potentials for these two nuclear models (neglecting the A^{-1} corrections) are presented in Table II, where we have written V^F and V^B for the Fermi gas model and the Brueckner model, respectively, in the form

$$V^F(\pi^{\pm}, q_0) = \text{Re } V^F + i \text{ Im } V^F,$$

$$V^B(\pi^{\pm}, q_0) = \text{Re } V^B + i \text{ Im } V^B, \qquad (4.24)$$

and we have assumed $\lambda = 1$.

Longo[16] has recently deduced from experiments with 3 Bev/c π^+ mesons on various nuclei the values for the imaginary parts of the optical potential listed in Table III. The real parts are small, and the nucleon density distributions used are those inferred from electron scattering experiments. The corresponding values of Im V^F and Im V^B of equation (4.24) (neglecting the A^{-1} corrections and adjusted to Longo's central density) have been listed in Table III for comparison. Also included is the result for the first-order potential alone.

TABLE II. The well-depths of the $\pi^{\pm}$-nucleus optical model potential for a Fermi gas model of the nucleus (V^F = Re V^F + i Im V^F) and a Brueckner model of the nucleus (V^B = Re V^B + i Im V^B). T_π is the barycentric kinetic energy of the pion. We have neglected the A^{-1} corrections.

T_π(Bev)	Re V^F(Mev)	Im V^F(Mev)	Re V^B(Mev)	Im V^B(Mev)
1.0	14.9	-49.2	16.6	-54.3
1.1	9.4	-47.3	10.8	-52.4
1.2	7.2	-52.6	8.8	-58.7
1.3	11.0	-53.4	13.4	-59.2
1.4	15.9	-51.7	19.1	-56.7
1.5	19.0	-48.5	22.5	-52.8
1.6	18.2	-44.3	21.2	-47.9
1.7	14.6	-43.1	17.0	-46.4
1.8	12.3	-42.4	14.3	-45.9
1.9	10.9	-41.9	12.6	-45.3
2.0	9.9	-41.6	11.5	-45.1
2.2	8.6	-41.9	9.9	-45.4
2.4	7.7	-42.3	8.9	-46.0
2.6	7.0	-42.5	8.1	-46.1
3.0	6.1	-42.6	7.1	-46.2
3.5	5.2	-42.6	6.1	-46.3
4.0	4.6	-42.7	5.4	-46.4
4.5	4.1	-42.7	4.8	-46.4
5.0	3.7	-42.7	4.3	-46.4

TABLE III. Comparison of theoretical and experimental values of the imaginary part of the pion-nucleus optical potential for π^+ with momenta 3 Bev/c. Experimental data are from reference 16.

Element	Im $V^{expt.}$ (Mev)	Im V^B(Mev)	Im V^F(Mev)	Im V_1(Mev)
Be^9	-154 ± 9	-171	-133	-117
C^{12}	-59.4 ± 4.0	-63.5	-57.4	-53
Al^{27}	-58.5 ± 4.1	-60.4	-54.7	-50.5
Cu	$-69\left\{\begin{matrix}+13.0\\-8.0\end{matrix}\right.$	-60.4	-54.7	-50.5

V. THE NUCLEON-NUCLEUS OPTICAL POTENTIAL

The nucleon-nucleus first-order optical potential has been evaluated in terms of nucleon-nucleon phase-shifts by several authors.[12-14] The highest energy at which a complete set of such phase shifts presently exists is 310 Mev. This energy is probably near the lower limit of validity of our approximate evaluation of the second-order optical potential V_2, so the most accurate numerical evaluation of V_2 that can be carried through is for 310 Mev incident nucleons. However, we shall attempt to estimate the order of magnitude of V_2 for other energies.

The validity of the multiple scattering equations (2.8) in the case of incident nucleons is not immediately evident, as the effect of the Pauli principle on the incident and target nucleons has not been properly considered. Takeda and Watson[9] have shown, however, that for high energy incident nucleons the effect of the Pauli principle is properly accounted for -- to a good approximation -- if one uses scattering operators t_α antisymmetrized between the incident and α^{th} nucleon only. But such scattering operators are precisely those describing the scattering from a free nucleon, so the analysis of Section II is valid to a good approximation in this case also.

Following the discussion in Section IV, we project the nucleon-nucleon scattering operator t_α onto the isospin substates corresponding to I = 1 and I = 0. We use the respective projection operators Λ_1 and Λ_0 to write

$$t_\alpha^0 = t_\alpha^0(1)\Lambda_1 + t_\alpha^0(0)\Lambda_0. \quad (5.1)$$

The scattering operators t_α^0 describe the scattering in the nucleon-nucleus barycentric system, whereas the phase-shift analyses determine the scattering amplitudes $f_{c\alpha}^0$ in the nucleon-nucleon center-of-momentum (CM) system. The relation between these quantities is given by equation (A-9) of Appendix A:

$$t_\alpha^0(1) = \frac{-1}{(2\pi)^2\epsilon_0}\left(\frac{q_0}{k_0}\right) f_{c\alpha}^0(1),$$

$$t_\alpha^0(0) = \frac{-1}{(2\pi)^2\epsilon_0}\left(\frac{q_0}{k_0}\right) f_{c\alpha}^0(0), \quad (5.2)$$

to first-order in the angle of scattering. Here q_0 and k_0 are the momentum in the barycentric and nucleon-nucleon CM systems, respectively.

In terms of $f_c(1)$ and $f_c(0)$, define

$$f \equiv \frac{1}{4}\,[3f_c^0(1) + f_c^0(0)],$$

$$f_\tau \equiv \frac{1}{4}\,[f_c^0(1) - f_c^0(0)]. \quad (5.3)$$

Then, from Appendix C,

$$(0|f_L^0|0) = (\frac{q_0}{k_0}) \left[(A_0 \pm \frac{2T_3}{A} A_\tau) + (C_0' \pm \frac{2T_3}{A} C_\tau') \, X(\vec{q},\vec{q}_0) \right] \tag{5.4}$$

where A_0, A_τ, C_0', and C_τ' -- defined in Appendix D -- are to be evaluated at zero angle of scattering, and $X(\vec{q},\vec{q}_0)$ is defined in equation (C-9).

The presence of the spin-dependent term in equation (5.4) makes it convenient to decompose V_1 into two terms, one spin-independent and the other spin-dependent. Accordingly, we rewrite equation (3.16) as

$$\mathcal{V}_1(\vec{x}) = \mathcal{U}_1(\vec{x}) + \mathcal{W}_1(\vec{x}) \, \vec{\sigma} \cdot \vec{L}, \tag{5.5}$$

where $\vec{L}$ is the orbital angular momentum operator of the incident particle. Evidently the spin-independent part, $\mathcal{U}_1(\vec{x})$, may be written as in equation (3.16):

$$\mathcal{U}_1(\vec{x}) = U_1(N^{\pm},q_0) \, \rho(\vec{x}),$$

$$U_1(N^{\pm},q_0) = -\frac{2\pi}{\epsilon_0} (\frac{A}{V_A}) (\frac{q_0}{k_0}) (A_0 \pm \frac{2T_3}{A} A_\tau), \tag{5.6}$$

where equation (5.4) has been used. Here $N^{\pm}$ refers to incident protons or neutrons, respectively.

As the spin-dependent term in equation (5.4) depends on the final momentum q to obtain $\mathcal{W}_1(\vec{x})$, the passage from the momentum representative of $\mathcal{H}_1$ to the coordinate representative -- as effected in equations (3.12) through (3.14) -- must be reconsidered. By comparison with equation (3.8) we write

$$< \vec{q}|\mathcal{H}_1|\vec{q}_0 > = \frac{-A}{(2\pi)^2\epsilon_0 V_A} \left(\frac{q_0}{k_0}\right) \left(C_0' \pm \frac{2T_3}{A} C_T'\right) \chi(\vec{q},\vec{q}_0) \int d^3z \rho(\vec{z}) e^{-i(\vec{q}-\vec{q}_0)\cdot\vec{z}} . \tag{5.7}$$

Using the definition of $\chi(\vec{q},\vec{q}_0)$ from equation (C-9) and the relation

$$\vec{q}\, e^{i\vec{q}\cdot\vec{x}} = \frac{1}{i} \vec{\nabla}_x\, e^{i\vec{q}\cdot\vec{x}}, \tag{5.8}$$

the $\vec{\nabla}_x$ may be taken outside the integral over $\vec{q}$ [see equation (3.12)], which may then be evaluated to give

$$< \vec{x}|\mathcal{H}_1|\vec{q}_0 > = - \frac{2\pi}{\epsilon_0} \left(\frac{A}{V_A}\right) \frac{1}{k_0^2} \left(C_0' \pm \frac{2T_3}{A} C_T'\right) \frac{1}{i} \vec{\sigma} \cdot (\vec{q}_0 \times \vec{\nabla}_x) \left[\rho(\vec{x}) \frac{e^{i\vec{q}_0\cdot\vec{x}}}{(2\pi)^{3/2}}\right] . \tag{5.9}$$

If $\rho(\vec{x}) = \rho(|\vec{x}|)$,

$$\vec{q}_0 \times \vec{\nabla}_x \left[\rho(\vec{x})\, e^{i\vec{q}_0\cdot\vec{x}}\right] = \frac{1}{x} \frac{d\rho}{dx} (\vec{q}_0 \times \vec{x})\, e^{i\vec{q}_0\cdot\vec{x}} . \tag{5.10}$$

Then, using

$$< \vec{x}|\mathcal{H}_1|\vec{q}_0 > = \int < \vec{x}|\mathcal{H}_1|\vec{x}' > d^3x' \frac{e^{i\vec{q}_0\cdot\vec{x}'}}{(2\pi)^{3/2}} ,$$

it follows that

$$< \vec{x}|\mathscr{H}_1|\vec{x}' > = \mathscr{H}_1(\vec{x})\ \delta(\vec{x} - \vec{x}')\ \vec{\sigma} \cdot \vec{L}, \tag{5.11}$$

where the angular momentum $\vec{L}$ has been identified: $\vec{L} = \vec{x} \times \vec{q}_0$. $\mathscr{H}_1(\vec{x})$ may now be written in the familiar form

$$\mathscr{H}_1(\vec{x}) = W_1(N^{\pm}, q_0)\ (\frac{1}{\mu})^2\ \frac{1}{x}\frac{d\rho}{dx}\ ,$$

$$W_1(N^{\pm}, q_0) = -\frac{2\pi}{\epsilon_0}\ (\frac{A}{V_A})\ (\frac{\mu}{k_0})^2\ (i\ C_0' \pm \frac{2T_3}{A}\ i\ C_\tau'), \tag{5.12}$$

where the factor μ is the pion rest mass ($\approx$ 140 Mev) and has been included in equations (5.12) so as to cause the dimensions of W_1 to be those of an energy.

We write $\mathscr{V}_2(\vec{x})$ as in equation (5.5):

$$\mathscr{V}_2(\vec{x}) = \mathscr{U}_2(\vec{x}) + \mathscr{H}_2(\vec{x})\ \vec{\sigma} \cdot \vec{L}$$

$$= U_2\rho(\vec{x}) + W_2\ \frac{1}{\mu^2}\ \frac{1}{x}\ \frac{d\rho}{dx}\,\vec{\sigma} \cdot \vec{L}. \tag{5.13}$$

The quantities S and S_τ of equation (3.39) are evaluated in Appendix C. In terms of these results

$$U_2(N^{\pm},q_0) = \frac{i(2\pi)^2}{2\epsilon_0 q_0} \left(\frac{A}{V_A}\right)^2 \left(\frac{q_0}{k_0}\right)^2 \Big\{ - (R_s + R_a) \left[(\overline{f})^2 - \frac{1}{A}(\overline{f^2} + 3\overline{f_\tau^2}) \pm \frac{4T_3}{A} \overline{f}\, \overline{f}_\tau \right]$$

$$+ (R_s - R_a) \left[\frac{1}{4}(\overline{f^2} + 3\overline{f_\tau^2}) - \frac{4}{A}(\overline{f})^2 \pm \frac{2T_3}{A} \overline{ff_\tau} \right] \Big\} , \tag{5.14}$$

where the desired spin-averages of f and f_τ are given in equation (C-13).

The spin-dependent part W_2 is obtained by a treatment analogous to that leading from equation (5.7) to equation (5.12), with the result

$$W_2(N^{\pm},q_0) = \frac{(2\pi)^2}{2\epsilon_0 k_0} \left(\frac{A}{V_A}\right)^2 \left(\frac{\mu}{k_0}\right)^2 \left[(R_s' + R_a')h_d + (R_s' - R_a')h_e \right] , \tag{5.15}$$

where h_d and h_e are defined in equation (C-14). R_s' and R_a' are related to the R_s and R_a of equation (3.34) through the equations

$$R_s' = R_s + \frac{1}{iq_0} G_s(0),$$

$$R_a' = R_a + \frac{1}{iq_0} G_a(0).$$

The additional terms come from a spin-dependent scattering in the intermediate state. They are thus of order $\Theta_{q'} \simeq (q_0 R_s)^{-1}$ and, in the spirit of the high energy evaluation of equation (3.33), they may be discarded. We thus set

$$R_s' \simeq R_s, \qquad R_a' \simeq R_a.$$

Using the expression for Δ from equation (C-15) (ignoring the spin-dependent terms, as they vanish for forward scattering), v of equation (3.46) is given as

$$v(\vec{x}) = U\,\rho(\vec{x}) + W\,\frac{1}{\mu^2}\,\frac{1}{x}\,\frac{d\rho}{dx}\,\vec{\sigma}\cdot\vec{L},$$

$$U(N^{\pm},q_0) = \left\{1 - \frac{1}{A}\left[1 + \frac{\overline{f^2} + 3\overline{f_\tau^2} - \overline{f}^2}{(\overline{f})^2}\right]\right\}\left[U_1(N^{\pm},q_0) + U_2(N^{\pm},q_0)\right],$$

$$W(N^{\pm},q_0) = \left\{1 - \frac{1}{A}\left[1 + \frac{\overline{f^2} + 3\overline{f_\tau^2} - \overline{f}^2}{(\overline{f})^2}\right]\right\}\left[W_1(N^{\pm},q_0) + W_2(N^{\pm},q_0)\right].$$

(5.16)

In Appendix C, the expressions for the f's and h's of equations (5.14) and (5.15) are given in terms of the parameters A, B, C, H of the nucleon-nucleon scattering amplitudes. Further, Appendix D gives expressions for these parameters in terms of the nucleon-nucleon phase shifts (for small angles of scattering).

Gammel and Thaler[35] have found a set of potentials which match the 310 Mev phase shifts of Stapp et al.[36] for p-p scattering and which also reproduce the n-p experimental data at the same energy. Kerman, McManus, and Thaler[14] have used the phase shifts deduced from these potentials to evaluate the quantities A, B, ..., H_τ for $\theta_c = 0^\circ \rightarrow 160^\circ$ (θ_c being the CM scattering angle) and for energies 90, 156, and 310 Mev. Using their results we obtain the values listed in Table IV for $\theta_c = 0$.

TABLE IV. The nucleon-nucleon scattering parameters as defined in Appendix D. The numerical values are from reference (14).

	90 Mev	156 Mev	310 Mev
A_0	0.592 + 0.444i	0.475 + 0.401i	0.139 + 0.479i
A_τ	-0.069 - 0.169i	-0.006 - 0.114i	0.179 - 0.113i
C'_0	0.0888 + 0.264i	0.109 + 0.418i	0.117 + 0.480i
C'_τ	-0.0544 + 0.020i	-0.092 - 0.011i	-0.072 - 0.044i
B_0	-0.0158 - 0.0053i	-0.0279 - 0.0164i	0.052 - 0.043i
B_τ	-0.254 - 0.0676i	-0.257 - 0.0182i	-0.219 + 0.022i
H_0	0.114 - 0.0737i	0.144 - 0.164i	0.160 - 0.145i
H_τ	0.143 - 0.030i	0.126 + 0.007i	0.128 + 0.005i

Equation (5.16) may now be evaluated. We write it in the form (to order A^{-1}):

$$U(N^{\pm},q_0) = \frac{1}{\lambda^3}\left[(U^R_{00} + \frac{1}{A}U^R_{01} \pm \frac{T_3}{A}U^R_{02}) + i(U^I_{00} + \frac{1}{A}U^I_{01} \pm \frac{T_3}{A}U^I_{02})\right]$$

$$+ \frac{1}{\lambda^6}\frac{(R_s + R_a)}{10^{-13}}\left[(U^R_{10} + \frac{1}{A}U^R_{11} \pm \frac{T_3}{A}U^R_{12}) + i(U^I_{10} + \frac{1}{A}U^I_{11} \pm \frac{T_3}{A}U^I_{12})\right]$$

$$+ \frac{1}{\lambda^6}\frac{(R_s - R_a)}{10^{-13}}\left[(U^R_{20} + \frac{1}{A}U^R_{21} \pm \frac{T_3}{A}U^R_{22}) + i(U^I_{20} + \frac{1}{A}U^I_{21} \pm \frac{T_3}{A}U^I_{22})\right] .$$

$$W(N^{\pm},q_0) = \frac{1}{\lambda^3}\left[(W^R_{00} + \frac{1}{A}W^R_{01} \pm \frac{T_3}{A}W^R_{02}) + i(W^I_{00} + \frac{1}{A}W^I_{01} \pm \frac{T_3}{A}W^I_{02})\right]$$

$$+ \frac{1}{\lambda^6}\frac{(R_s + R_a)}{10^{-13}}\left[(W^R_{10} + \frac{1}{A}W^R_{11} \pm \frac{T_3}{A}W^R_{12}) + i(W^I_{10} + \frac{1}{A}W^I_{11} \pm \frac{T_3}{A}W^I_{12})\right]$$

$$+ \frac{1}{\lambda^6}\frac{(R_s - R_a)}{10^{-13}}\left[(W^R_{20} + \frac{1}{A}W^R_{21} \pm \frac{T_3}{A}W^R_{22}) + i(W^I_{20} + \frac{1}{A}W^I_{21} \pm \frac{T_3}{A}W^I_{22})\right] . \tag{5.17}$$

Here λ is defined by equation (4.13). The quantities appearing in equation (5.17) are listed in Table V for incident nucleons of 90, 156, and 310 Mev kinetic energy.

For a Fermi gas model of the nucleus, we use equations (4.19) in equations (5.14) and (5.15) to obtain

TABLE V. The parameters of the nucleon-nucleus optical potential as defined by equations (5.17). T_p is the nucleon kinetic energy in the nucleon-nucleus barycentric system, and R_s and R_a are measured in centimeters. (f = fermi = 10^{-13} cm.)

T_p(Mev)	U^R_{00}(Mev)	U^R_{01}(Mev)	U^R_{02}(Mev)	U^I_{00}(Mev)	U^I_{01}(Mev)	U^I_{02}(Mev)
90	-39.3	(-1545)	9.31	-29.9	(-1130)	22.8
156	-30.6	98.8	0.77	-25.8	-22.3	14.7
310	-8.1	3.7	-21.	-28.	-59.	14.

	U^R_{10}(Mev f^{-1})	U^R_{11}(Mev f^{-1})	U^R_{12}(Mev f^{-1})	U^I_{10}(Mev f^{-1})	U^I_{11}(Mev f^{-1})	U^I_{12}(Mev f^{-1})
90	0.66	-24.9	-14.8	-0.67	26.0	-3.86
156	7.75	-9.15	-4.59	-1.32	17.5	-3.47
310	1.8	1.5	3.7	2.8	6.5	-4.2

	U^R_{20}(Mev f^{-1})	U^R_{21}(Mev f^{-1})	U^R_{22}(Mev f^{-1})	U^I_{20}(Mev f^{-1})	U^I_{21}(Mev f^{-1})	U^I_{22}(Mev f^{-1})
90	-6.2	2.6	8.35	6.5	-2.7	5.41
156	-2.29	31.0	4.06	4.37	-5.28	5.12
310	.37	7.0	-1.4	1.6	11.	2.5

	W^R_{00}(Mev)	W^R_{01}(Mev)	W^R_{02}(Mev)	W^I_{00}(Mev)	W^I_{01}(Mev)	W^I_{02}(Mev)
90	4.12	157.	0.62	-1.40	-64.	1.7
156	4.49	-3.47	-0.18	-0.91	8.43	1.54
310	1.8	3.7	-0.32	-0.43	0.40	0.53

TABLE V (cont.)

T_p(Mev)	W^R_{10}(Mev f^{-1})	W^R_{11}(Mev f^{-1})	W^R_{12}(Mev f^{-1})	W^I_{10}(Mev f^{-1})	W^I_{11}(Mev f^{-1})	W^I_{12}(Mev f^{-1})
90	-0.42	0	0	1.95	-1.37	-1.46
156	-0.30	0.12	-0.04	0.63	-0.72	-0.54
310	-0.18	0.16	0.14	0.10	-0.14	0.05

	W^R_{20}(Mev f^{-1})	W^R_{21}(Mev f^{-1})	W^R_{22}(Mev f^{-1})	W^I_{20}(Mev f^{-1})	W^I_{21}(Mev f^{-1})	W^I_{22}(Mev f^{-1})
90	0	-1.68	0.04	-0.34	7.8	0.77
156	0.03	-1.2	0.03	-0.18	2.5	0.43
310	0.04	-0.71	-0.04	-0.04	0.41	0.06

$$U_2^F = i\,\frac{2\pi}{\epsilon_0 q_0}\;\frac{k_F^2}{5}\,\left(\frac{A}{V_A}\right)\left(\frac{q_0}{k_0}\right)^2\left(\overline{f^2} + 3\overline{f_\tau^2} \pm \frac{8T_3}{A}\,\overline{ff_\tau}\right)\left(1 - \frac{4}{A}\right)\lambda^{-5},$$

$$W_2^F = \frac{2\pi}{\epsilon_0 k_0}\;\frac{k_F^2}{5}\;\left(\frac{A}{V_A}\right)\left(\frac{\mu}{k_0}\right)^2\left\{\left[(A_0 + B_0)C_0' + 3(A_\tau + B_\tau)C_\tau'\right]\right.$$

$$\left.\pm\,\frac{4T_3}{A}\left[(A_0 + B_0)C_\tau' + (A_\tau + B_\tau)C_0'\right]\right\}\left(1 - \frac{4}{A}\right)\lambda^{-5}. \qquad (5.18)$$

The numerical evaluation of equation (5.18) gives (neglecting the A^{-1} corrections):

$$T_p = 90 \text{ Mev:}\quad \mathcal{V}^F(x) = (-58 - 11i)\,\rho(x) + (4.1 - 2.4i)\,\frac{1}{\mu^2}\,\frac{1}{x}\,\frac{d\rho}{dx}\,\vec{\sigma}\cdot\vec{L}\,;$$

$$T_p = 156 \text{ Mev:}\quad \mathcal{V}^F(x) = (-37 - 13i)\,\rho(x) + (4.6 - 1.5i)\,\frac{1}{\mu^2}\,\frac{1}{x}\,\frac{d\rho}{dx}\,\vec{\sigma}\cdot\vec{L}\,;$$

$$T_p = 310 \text{ Mev:}\quad \mathcal{V}^F(x) = (-7 - 23i)\,\rho(x) + (1.9 - 0.53i)\,\frac{1}{\mu^2}\,\frac{1}{x}\,\frac{d\rho}{dx}\,\vec{\sigma}\cdot\vec{L}\,, \qquad (5.19)$$

where we have assumed $\lambda = 1$, and T_p is the nucleon kinetic energy in the barycentric system.

Using the values of R_s and R_a from equation (4.23), deduced from the work of Brueckner and Gammel,[34] we obtain (again assuming $\lambda = 1$ and neglecting terms of order A^{-1}),

$$T_p = 90 \text{ Mev: } \mathcal{V}^B(x) = (-40 - 29i)\, \rho(x) + (4.8 - 4.7i)\, \frac{1}{\mu^2} \frac{1}{x} \frac{d\rho}{dx} \vec{\sigma} \cdot \vec{L}\, ;$$

$$T_p = 156 \text{ Mev: } \mathcal{V}^B(x) = (-44 - 24i)\, \rho(x) + (5.0 - 1.0i)\, \frac{1}{\mu^2} \frac{1}{x} \frac{d\rho}{dx} \vec{\sigma} \cdot \vec{L}\, ;$$

$$T_p = 310 \text{ Mev: } \mathcal{V}^B(x) = (-11 - 33i)\, \rho(x) + (2.1 - 0.6i)\, \frac{1}{\mu^2} \frac{1}{x} \frac{d\rho}{dx} \vec{\sigma} \cdot \vec{L}\, .$$

(5.20)

Batty[37] has analyzed the 310 Mev data of Chamberlain et al.[38] on C^{12}, treating carefully the coulomb effects. Using a Gaussian charge distribution for simplicity, and a modified Gaussian nucleon distribution of the form

$$\rho(x) = \frac{V_A}{3\pi^{3/2}\, a^3} \left(1 + \frac{4}{3} \frac{x^2}{a^2}\right) e^{-(x^2/a^2)}, \quad a = 1.635 \text{ fermi},$$

(5.21)

which gives the best fit to the electron scattering data on carbon,[39] he obtains

$$\mathcal{V}(x) = (-10.5 - 29.9i)\, \rho(x) + (2.68 - 0.32i)\, \frac{1}{\mu^2} \frac{1}{x} \frac{d\rho}{dx} \vec{\sigma} \cdot \vec{L}.$$

(5.22)

In the absence of phase shift analyses of nucleon-nucleon scattering at higher energies, we may try to estimate the magnitude of the second-order potential U_2 as follows. If U_2 is assumed to be small, the phenomenological optical potential $\mathcal{V}_0$ deduced from experiment is approximately equal to U_1. From equation (5.6), if the corrections of order A^{-1} are disregarded, A_0 may then be obtained from $\mathcal{V}_0$, and, for a Brueckner nuclear model, from equation (5.14) we see that U_2 depends only on A_0. Thus, for this nuclear model, we can obtain an estimate of U_2. In our results below we normalize the experimental $\mathcal{V}_0$ to correspond to a central nucleon density such that $\lambda = 1$.

Nedzel[40] has measured total cross sections for 410 Mev neutrons on a range of elements. He assumes Re $\mathcal{V}_0 = 0$, and he finds that, to obtain R_A proportional to $A^{1/3}$,

$$R_A = 1.23\, A^{1/3} \times 10^{-13} \text{ cm},$$

$$\text{Im } \mathcal{V}_0 \cong -25 \text{ Mev}.$$

Then Im $A_0 \sim 1.1$ fermi, and Im $U_2 \cong 1.3$ Mev.

Booth, Hutchinson, and Ledley[41] have fit their data on 765 Mev neutrons scattered from several nuclei to optical potentials with nucleon density distributions taken from electron scattering experiments. They assume the Re $\mathcal{V}_0$ vanishes and the spin-orbit potential is purely real. Their results, together with our estimates of Im U_2, are presented in Table VI(a).

Booth, Ledley, Walker, and White[42] have measured the total and differential cross sections for 900 Mev protons on C, Al, and Cu.

TABLE VI. The phenomenological optical potentials deduced by Booth et al. and the resulting estimates of the second-order potential. (a) 765 Mev neutrons [reference (41)]; (b) 900 Mev protons [reference (42)].

	Element	Im $\mathcal{V}_0$(Mev)	A_{0I}($\times 10^{-13}$cm)	Im U_2(Mev)
(a)	C^{12}	-43	2.2	4.9
	Cu	-45	2.2	4.9
	Pb	-45	2.2	4.9
(b)	C^{12}	-36	2.0	5.2
	Al^{27}	-52	2.9	11.0
	Cu	-45	2.5	8.1
	Sb	-46	2.6	8.8

Assuming a rectangular density distribution with $R_A = 1.26\ A^{1/3} \times 10^{-13}$ cm, they have determined the potential strength which best fits their data. Assuming the real part of $\mathcal{V}_0$ is small, their results and our estimates of U_2 are given in Table VI(b).

Coor, Hill, Hornyak, Smith, and Snow[43] have performed a similar analysis of their data on the scattering of 1.4 Bev neutrons from several nuclei ranging from Be and C to Pb, Bi, and U. They measured the total, absorption, and diffraction cross sections and found they can be equally well fit to within the experimental accuracy by either a rectangular or a Gaussian nucleon density distribution. With a rectangular well, they find a good fit to all their data with $R_a = 1.28\ A^{1/3} \times 10^{-13}$ cm, Re $\mathcal{V}_0$ small, and Im $\mathcal{V}_0 = -44$ Mev. Normalizing to $\lambda = 1$, this corresponds to Im $\mathcal{V}_0 = -54$ Mev. Then Im $A_0 \cong 3.7$ fermi and Im $U_2 \cong 9$ Mev.

Longo[16] has performed a careful optical model analysis of his data on the elastic scattering of protons with momenta 3 Bev/c from several nuclei. If Re $\mathcal{V}_0$ is taken to be small, he obtains a good fit to his data with Im $\mathcal{V}_0 \simeq -63$ Mev. Using the values $\sigma_{np} = 40$ mb, $\sigma_{pp} = 42$ mb, quoted by Longo, we obtain a first-order potential Im $U_1 = -54$ Mev. Attributing the difference between $\mathcal{V}_0$ and U_1 to the second-order potential, we find Im $U_2 \simeq -9$ Mev.

In general, these experimental results seem to be consistent with the assumption that the real part of the optical potential is quite small. If we assume nucleon-nucleon interactions to be purely inelastic and spin-independent -- for nucleons of kinetic energy somewhat greater than 700 Mev -- of the six parameters describing forward-scattering, only Im A is nonvanishing. Since Im A may be related directly to measured total cross sections, we may evaluate the corresponding second-order imaginary potential.

In Figure 1 are presented the results of such an evaluation -- for nucleon energies 700 Mev to 3 Bev -- for both a Fermi gas model and a Brueckner model of the nucleus. Included are the values from equations (5.19) and (5.20) above and the first-order imaginary potential -- covering the energy range 100 Mev to 3 Bev. The required total nucleon-nucleon cross sections are taken from the review article by Hess.[44] The experimental values of the imaginary part of the optical potential described above are included, as well as the values recently deduced by Batty[37] from the scattering of 420, 635, and 970 Mev protons from C^{12}.

In summary, we see that the contribution of the second-order optical potential to $\mathcal{V}_0$ constitutes a correction of 10 to 15% of the first-order potential, even at the highest energies considered. This correction may be reasonably well accounted for by utilizing equation (4.22) -- with $f_R = 0$ -- giving the dashed curve in Figure 1.

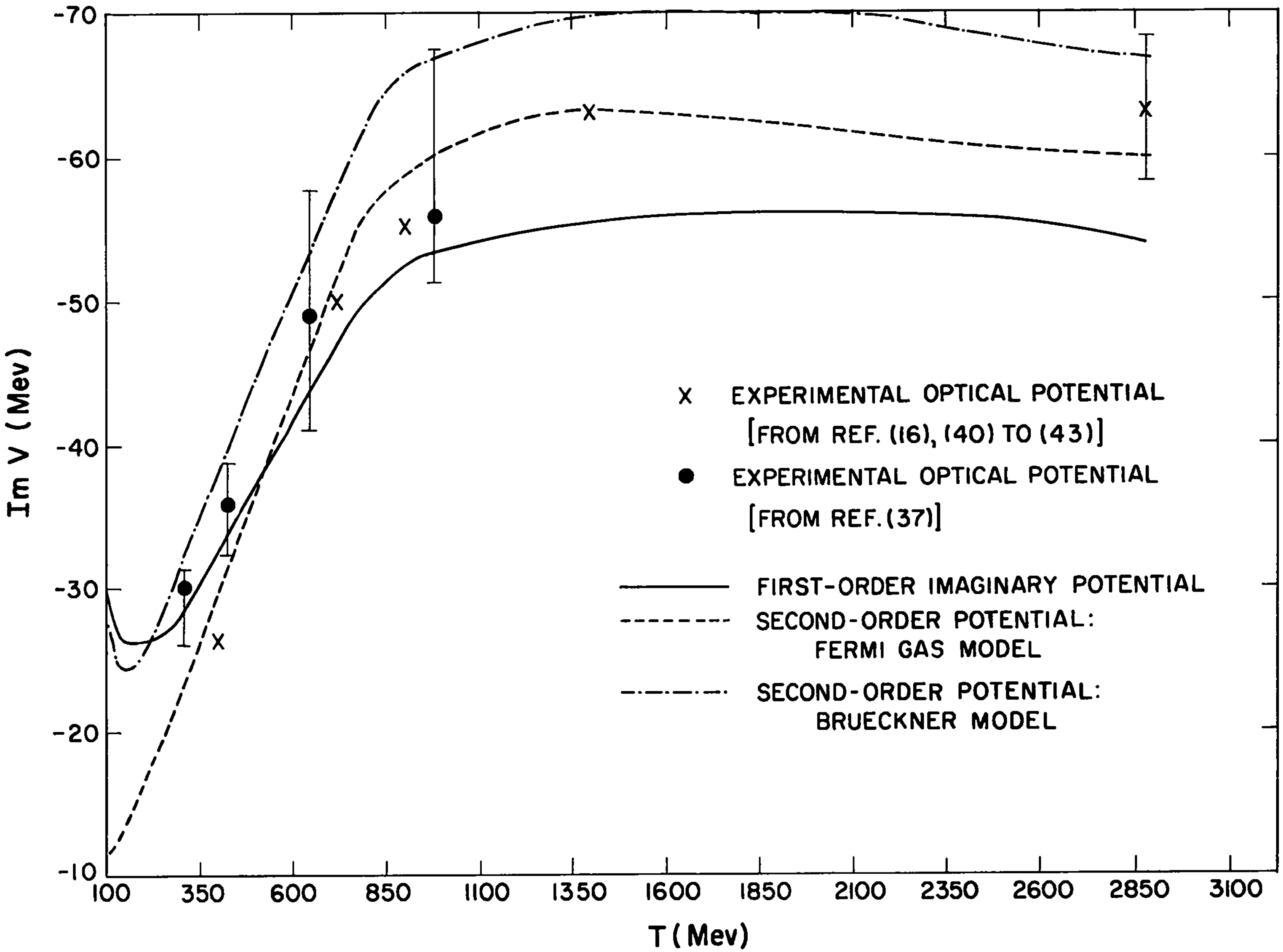

Figure 1. The imaginary part of the nucleon-nucleus central optical potential.

VI. NONLOCALITY OF THE OPTICAL POTENTIAL

The assumptions leading to equation (3.11) allowed us to obtain the expressions (3.16) and (3.39) for an optical potential local in coordinate representation. In this section we propose to examine the leading corrections to equation (3.16) due to the dependence of $t(\vec{q},\vec{q}_0)$ on the scattering angle. We shall find that the inclusion of such corrections leads to an optical potential nonlocal in coordinate space, but which may be written as a local potential with a nucleon density distribution modified from that obtained by other means -- for example, electron scattering.

This modification of the density distribution has frequently[15,20] been described as the inclusion of the effects of the finite range of interaction of the incident particle with nucleons of the target nucleus. As long as the energies are high enough that the free nucleon scattering operators are applicable, the effect of the finite range of interaction is properly included in these scattering operators. Therefore, the modification of the nucleon density is more correctly described as being a manifestation of the nonlocality of the optical potential.

In general, the scattering amplitudes in the two-body CM system for small angles of scattering θ_c may be written in the form

$$f_c(\vec{k},\vec{k}_0) = f_c^0(k_0) + f_c^{(1)}(k_0)\theta_c^2 + O(\theta_c^4). \qquad (6.1)$$

An explicit expression of this form for nucleon-nucleon scattering is given by equation (D-3) of Appendix D.

Employing the general relation between laboratory and CM scattering angles -- for incident nucleons --

$$2 \tan \frac{\theta_c}{2} = \frac{q_0}{k_0} \tan\theta_L , \qquad (6.2)$$

where k_0 is the CM momentum, we see

$$\theta_c^2 = \left(\frac{q_0}{k_0}\right)^2 \theta_L^2 + O(\theta_L^4). \qquad (6.3)$$

Thus we may write for the scattering amplitude in the laboratory system

$$f_L(\vec{q},\vec{q}_0) = f_L^0(q_0)(1 - \alpha\, q_0^2\theta_L^2) + O(\theta_L^4), \qquad (6.4)$$

where, comparing with equation (6.1),

$$\alpha = - \frac{1}{k_0^2} \frac{f_c^{(1)}(k_0)}{f_c^0(k_0)} . \qquad (6.5)$$

For incident particles other than nucleons, equation (6.4) is still valid, but the definition of α in equation (6.5) is modified.

Writing

$$\left(0\middle|t(\vec{q},\vec{q}_0)\middle|0\right) = \left(0\middle|t^0(q_0)\middle|0\right)(1 - \alpha\kappa^2), \qquad (6.6)$$

where

$$\vec{\kappa} = \vec{q}_0 - \vec{q},$$

$$\kappa^2 = \left(2q_0 \sin\frac{\theta_L}{2}\right)^2 = q_0^2\theta_L^2 + O(\theta_L^4), \tag{6.7}$$

equation (3.12) becomes, using equations (3.15) and (6.6),

$$\langle\vec{x}|\mathcal{V}_1|\vec{q}_0\rangle = \frac{-1}{(2\pi)^2\epsilon_0}\left(0|f_L^0(q_0)|0\right)\frac{A}{V_A}\frac{e^{i\vec{q}_0\cdot\vec{x}}}{(2\pi)^{3/2}}\int d^3\kappa(1 - \alpha\kappa^2)e^{-i\vec{\kappa}\cdot\vec{x}}d^3z\rho(\vec{z})e^{i\vec{\kappa}\cdot\vec{z}}. \tag{6.8}$$

Introducing the relation

$$\kappa^2 e^{-i\vec{\kappa}\cdot\vec{x}} = -\nabla_x^2 e^{-i\vec{\kappa}\cdot\vec{x}}$$

into equation (6.8), we obtain

$$\langle\vec{x}|\mathcal{V}_1|\vec{q}_0\rangle = \frac{-1}{(2\pi)^2\epsilon_0}\left(0|f_L^0(q_0)|0\right)\frac{A}{V_A}\frac{e^{i\vec{q}_0\cdot\vec{x}}}{(2\pi)^{3/2}}(2\pi)^3\left[\rho(\vec{x}) + \alpha\nabla^2\rho(\vec{x})\right]. \tag{6.9}$$

But, in general,

$$\langle\vec{x}|\mathcal{V}_1|\vec{q}_0\rangle = \int\langle\vec{x}|\mathcal{V}_1|\vec{x}'\rangle d^3x'\frac{e^{i\vec{q}_0\cdot\vec{x}'}}{(2\pi)^{3/2}}, \tag{6.10}$$

so

$$\langle\vec{x}|\mathcal{V}_1|\vec{x}'\rangle = \mathcal{V}_1(\vec{x})\delta(\vec{x} - \vec{x}') \tag{6.11}$$

with

$$\mathcal{V}_1(\vec{x}) = \frac{-2\pi}{\epsilon_0}\left(\frac{A}{V_A}\right)\left(0\right|\ f_L^0(q_0)\left|0\right)\left[\rho(\vec{x}) + \alpha\nabla^2\rho(\vec{x})\right] . \tag{6.12}$$

Comparing this expression with equation (3.16), we see that the effect of including the angle-dependence of t for small angles leads to a modified density distribution of the form

$$\bar{\rho}(\vec{x}) = \rho(\vec{x}) + \alpha\nabla^2\rho(\vec{x}). \tag{6.13}$$

Note that since the angle-dependence of the real and imaginary parts of t may be different, α will be complex and thus the effective density distributions of the real and imaginary parts of the optical potential will be different.

An instructive example is provided by considering a Gaussian density distribution

$$\rho(x) = \frac{V_A}{\pi^{3/2}a^3}\, e^{-x^2/a^2}. \tag{6.14}$$

Then

$$\bar{\rho}(x) = \frac{V_A}{\pi^{3/2}a^3}\, e^{-x^2/a^2}\left[1 - \frac{4\alpha}{a^2}\left(\frac{3}{2} - \frac{x^2}{a^2}\right)\right].$$

But consider

$$\rho'(x) = \frac{V_A}{\pi^{3/2} a^3 (1+\lambda)^{3/2}} \; e^{-x^2/a^2(1+\lambda)} .$$

For λ small,

$$\rho'(x) \simeq \frac{V_A}{\pi^{3/2} a^3} \; e^{-x^2/a^2} \left[1 - \lambda\left(\frac{3}{2} - \frac{x^2}{a^2}\right)\right],$$

so $\rho' = \overline{\rho}$ if we identify $\lambda = 4\alpha/a^2$. Therefore, the effect of considering the angle dependence of the scattering operator is to increase the rms radius of the density distribution from $\left[(3/2)a^2\right]^{1/2}$ to

$$\langle r' \rangle = \left[\frac{3}{2}\,(a^2 + 4\alpha)\right]^{1/2} . \qquad (6.15)$$

For the nucleon-nucleus spin-dependent optical potential, a result similar to equation (6.12) is obtained. Using equation (6.6) in equation (5.7), the integral term becomes

$$\int d^3q \, \frac{e^{i\vec{q}\cdot\vec{x}}}{(2\pi)^{3/2}} \, (1 - \beta\kappa^2)\vec{\sigma} \cdot (\vec{q}_0 x \vec{q}) d^3z \rho(\vec{z}) e^{i\vec{\kappa}\cdot\vec{z}} . \qquad (6.16)$$

Here β is the quantity for the spin-dependent scattering corresponding to the α of equation (6.8). Using equation (5.8), equation (6.16) becomes

$$\frac{1}{i}\, \vec{\sigma} \cdot (\vec{q}_0 x \vec{\nabla}_x) \int d^3q \, \frac{e^{i\vec{q}\cdot\vec{x}}}{(2\pi)^{3/2}} \, (1 - \beta\kappa^2) d^3z \rho(\vec{z}) e^{i\vec{\kappa}\cdot\vec{z}} . \qquad (6.17)$$

Algebraic manipulations similar to those leading from equation (6.8) to (6.12) and to those leading from equation (5.9) to (5.12) lead to the results

$$\mathscr{W}_1(\vec{x}) = \left[- \frac{2\pi}{\epsilon_0} \left(\frac{A}{V_A}\right) \left(\frac{\mu}{k_0}\right)^2 \left(iC_0' \pm \frac{2T_3}{A}\, iC_\tau' \right)\right] \left(\frac{1}{\mu^2}\right) \frac{1}{x} \frac{d}{dx} \left[\rho(x) + \beta\nabla^2\rho(x)\right]. \quad (6.18)$$

Comparing with equation (5.12) one sees that again the angle-dependence of the spin-dependent scattering amplitudes leads to a modification of the spatial dependence of the spin-orbit optical potential -- a modification which is different for the real and imaginary parts. Cromer[45] has recently obtained results similar to equations (6.12) and (6.18), but specialized to the case of a Gaussian distribution.

In Table VII are listed the values of A and C for 310 Mev nucleons for a range of small angles. Using these, the following values of α and β are obtained:

$$\alpha = 1.2 + 0.29i; \qquad \alpha_\tau = 0.62 + 0.37i;$$
$$\beta = 0.15i; \qquad \beta_\tau = 1.2 + 0.23i. \quad (6.19)$$

The analysis by Fregeau[39] of the scattering of electrons from carbon shows that for $\kappa^2 \leq 1.5$ ($\theta_L \leq 17^\circ$) the Gaussian density distribution, equation (6.14), with a = 1.96 fermi provides a good fit to the data. This corresponds to a rms radius $< r > = 2.4$ fermi, and we write it $\rho(x| < r >) = \rho(x|2.4)$. Using the results of equation (6.19) in (6.15),

Table VII. Angle-dependence of the nucleon-nucleon scattering parameters for 310 Mev nucleons. Data are from reference (14).

θ_c	Re $A_0(f)$	Im $A_0(f)$	Re $A_\tau(f)$	Im $A_\tau(f)$
2^o	0.139	0.479	0.179	-0.1130
4^o	0.136	0.477	0.177	-0.1125
6^o	0.132	0.474	0.175	-0.111
8^o	0.127	0.469	0.171	-0.110

θ_c	Re $C'_0(f)$	Im $C'_0(f)$	Re $C'_\tau(f)$	Im $C'_\tau(f)$
2^o	0.117	0.480	-0.2865	-0.175
4^o	0.117	0.479	-0.2853	-0.171
6^o	0.1167	0.480	-0.2841	-0.166
8^o	0.1157	0.481	-0.2817	-0.160

the optical potential for the scattering of 310 Mev nucleons from carbon -- equations (5.6) and (5.12) -- may be written

$$\mathcal{V}_1(x) = \mathrm{Re}\ U_1\rho(x|3.6) + i\ \mathrm{Im}\ U_1\rho(x|3.1)$$

$$+ \frac{1}{\mu^2}\frac{1}{x}\frac{d}{dx}\left[\mathrm{Re}\ W_1\rho(x|2.4) + i\ \mathrm{Im}\ W_1\rho(x|3.6)\right]. \qquad (6.20)$$

Kisslinger[46] has applied similar considerations to pion-nucleus scattering. He is concerned, however, with pions of lower energy such that only S- and P-waves are expected to be important. Thus, he writes

$$t(\vec{q},\vec{q}_0) = a(q_0) + b(q_0)\cos\theta \qquad (6.21)$$

as valid for all θ, not just small angles. Using equation (6.21) -- with $\cos\theta = (1/q_0^2)\vec{q}\cdot\vec{q}_0$ -- in equation (6.8) and carrying through similar manipulations, we obtain his result

$$v(\vec{x}) = (2\pi)^3\left(\frac{A}{V_A}\right)\left[a\rho(\vec{x}) - b(\vec{\nabla}\rho)\cdot\vec{\nabla}\right]. \qquad (6.22)$$

Baker, Byfield, and Rainwater[47] have found that to obtain large-angle agreement with their experiments at 80 Mev, a potential of the form (6.22) is necessary.

From comparison of equations (3.16) and (3.39) we see

$$\frac{V_2}{V_1} \approx \frac{1}{q_0 R_A}. \qquad (6.23)$$

Moreover, the above modifications of the nucleon density distributions are proportional to $\theta^2 \approx (q_0R_A)^{-2}$, by equation (3.10). So consideration of terms of order θ^2 in V_2 will lead to corrections of order $(q_0R_A)^{-3}$, which we have agreed to neglect [(see equation (3.47)].

APPENDIX A

RELATIVISTIC KINEMATICS

The following basic relation between scattering operators and differential cross sections has been given by Møller:[48]

$$d\sigma = \frac{(2\pi)^4 \, k^2 \left| < \vec{k},\vec{P} | t | \vec{k}_0,\vec{P}_0 > \right|^2}{\left(|\vec{u}_1^{\,0} - \vec{u}_2^{\,0}|^2 - |\vec{u}_1^{\,0} \times \vec{u}_2^{\,0}|^2 \right)^{1/2} \left| (\vec{u}_2^{\,1} - \vec{u}_1^{\,1}) \cdot \hat{k} \right|} \, d\Omega = |f(\theta)|^2 \, d\Omega \tag{A-1}$$

for scattering into an element of solid angle $d\Omega$ about $\vec{k}$ of a particle with initial momentum $\vec{k}_0$, velocity $\vec{u}_1^{\,0}$, on a target of momentum $\vec{P}_0$, velocity $\vec{u}_2^{\,0}$. $\vec{u}_1^{\,1}$ and $\vec{u}_2^{\,1}$ are the corresponding velocities after scattering. Møller has also shown that the quantity

$$\sqrt{\epsilon' E'} < \vec{k},\vec{P} | t | \vec{k}_0,\vec{P}_0 > \sqrt{\epsilon E} \tag{A-2}$$

is Lorentz-invariant, ϵ and E being the energies of the particle and target, respectively.

We are concerned here with three coordinate systems: the laboratory system, the particle-nucleus barycentric system, and the particle-nucleon CM system.

In the particle-nucleon CM system:

$$\vec{k}_0 = -\vec{P}_0, \quad \vec{k} = -\vec{P}, \quad |\vec{k}| = |\vec{k}_0|,$$

$$\vec{u}_1^{\,0} = \frac{\vec{k}_0}{\epsilon_c}, \quad \vec{u}_2^{\,0} = \frac{\vec{P}}{E_c}.$$

Then equation (A-1) gives

$$f_c(\theta) = -(2\pi)^2\, t_c \left(\frac{\epsilon_c E_c}{\epsilon_c + E_c}\right). \qquad \text{(A-3)}$$

Similarly, in the laboratory system (for forward scattering), $\vec{u}_2^{\,0} = \vec{u}_2^{\,1} = 0$.

$$f_L^0 = -(2\pi)^2\, t_L \epsilon_L. \qquad \text{(A-4)}$$

Using equation (A-2) we find, for forward scattering,

$$t_L^0 = \frac{-1}{(2\pi)^2\, \epsilon_L} f_L^0 = \frac{-1}{(2\pi)^2\, \epsilon_L} \left(\frac{k_L}{k_c}\right) f_c^0. \qquad \text{(A-5)}$$

In the particle-nucleus barycentric system:

$$\vec{u}_1^{\,0} = \frac{\vec{k}_0}{\epsilon_B}, \qquad \vec{u}_2^{\,0} = \frac{\vec{P}_0}{E_B}, \qquad \vec{P}_0 = -\frac{1}{A}\vec{k}_0.$$

Then, for forward scattering,

$$f_B^0 = -(2\pi)^2 \, t_B^0 \left(\frac{\epsilon_B E_B}{E_B + \frac{1}{A} \epsilon_B} \right) \tag{A-6}$$

Expressing f_B in terms of ϵ_L and using (A-2), we obtain (discarding terms of order A^{-2})

$$f_B^0 \simeq \left(1 + \frac{2\epsilon_B}{AM}\right)^{-1/2} f_L^0, \tag{A-7}$$

where M is the nucleon mass.

Combining equations (A-6) and (A-7) gives

$$t_B^0 = \frac{-1}{(2\pi)^2 \, \epsilon_B} \, f^0{}_L \left(\frac{\sqrt{1 + \frac{2\epsilon_L}{AM}}}{1 + \frac{\epsilon_L}{AM}} \right) \tag{A-8}$$

Thus, if we may neglect terms of order $(\epsilon_L/AM)^2$,

$$t_B^0 = \frac{-1}{(2\pi)^2 \, \epsilon_B} \, f_L^0 = \frac{-1}{(2\pi)^2 \, \epsilon_B} \left(\frac{q_0}{k_0}\right) f_c^0, \tag{A-9}$$

where q_0 and k_0 are, respectively, the momentum of the incident particle in the laboratory and particle-nucleon CM system.

APPENDIX B

EVALUATION OF THE PION-NUCLEUS SPIN-ISOSPIN AVERAGES

By the hypothesis of charge-independence, the scattering matrices must be invariant under rotations in isospin space; that is, t_α^0 must have isospin dependence only of the form $\vec{I}\cdot\vec{I}$, where $\vec{I}$ is the total isospin of the pion-nucleon system. We introduce[49] creation and annihilation operators U_i, U_j $(i,j = 1,2,3)$ for the three types of pions ω_1, ω_2, ω_3.

$$\vec{U} = \omega_1\hat{e}_1 + \omega_2\hat{e}_2 + \omega_3\hat{e}_3. \tag{B-1}$$

It is easy to show that the quantity

$$\vec{L} = i\,\vec{U}\times\vec{U}^\dagger \tag{B-2}$$

transforms like a vector in isospin space, so we choose

$$\vec{I} = \frac{1}{2}\vec{\tau}_\alpha + i\,\vec{U}\times\vec{U}^\dagger, \tag{B-3}$$

where $\vec{\tau}_\alpha$ is the isospin operator of the α^{th} nucleon.

The most general form of t_α^0 is then

$$t_\alpha^0 = A_0 \vec{U}^\dagger \cdot \vec{U} + i\, B_0 \vec{U}^\dagger \times \vec{U} \cdot \vec{\tau}_\alpha, \qquad \text{(B-4)}$$

where A_0 and B_0 are independent of the nucleon spin for forward scattering. One can verify the projection operators

$$\Lambda_{3/2} = \frac{1}{3}\,(2\vec{U}^\dagger \cdot \vec{U} - i\vec{U}^\dagger \times \vec{U} \cdot \vec{\tau}),$$

$$\Lambda_{1/2} = \frac{1}{3}\,(\vec{U}^\dagger \cdot \vec{U} + i\vec{U}^\dagger \times \vec{U} \cdot \vec{\tau}), \qquad \text{(B-5)}$$

allowing the identification of A_0 and B_0 with the two independent scattering matrices

$$t_\alpha^0 = (A_0 - B_0)\Lambda_{3/2} + (A_0 + 2B_0)\Lambda_{1/2} = t_\alpha^0\,(\tfrac{3}{2})\Lambda_{3/2} + t_\alpha^0\,(\tfrac{1}{2})\Lambda_{1/2}.$$

$$\text{(B-6)}$$

Since we assume only one pion to be present in the intermediate state, we find

$$t_\alpha^0 t_\beta^0 = A_0^2\, \vec{U}^\dagger \cdot \vec{U} + iA_0B_0\, \vec{U}^\dagger \times \vec{U} \cdot (\vec{\tau}_\alpha + \vec{\tau}_\beta)$$

$$- B_0^2 \left[(\vec{\tau}_\beta \cdot \vec{U}^\dagger)(\vec{\tau}_\alpha \cdot \vec{U}) - \vec{\tau}_\alpha \cdot \vec{\tau}_\beta\, \vec{U}^\dagger \cdot \vec{U} \right]. \qquad \text{(B-7)}$$

Using the results of Table VIII -- keeping terms of order A only -- we find (for unpolarized nuclei)

$$\sum_{\alpha\neq\beta=1}^{A}(0|t_\alpha^0 t_\beta^0|0) = \vec{U}^\dagger\cdot\vec{U}\,[A(A-1)A_0^2 - 2AB_0^2] + 4iAA_0B_0\vec{U}^\dagger\times\vec{U}\cdot\hat{e}_3T_3,$$

$$\sum_{\alpha\neq\beta=1}^{A}(0|t_\alpha^0 t_\beta^0 P_{\alpha\beta}|0) = \vec{U}^\dagger\cdot\vec{U}\,[-\tfrac{1}{4}A(A-16) - \tfrac{1}{2}A^2B_0^2] - 2iAA_0B_0\vec{U}^\dagger\times\vec{U}\cdot\hat{e}_3T_3,$$

$$\sum_{\alpha=1}^{A}(0|t_\alpha^0|0) = AA_0\vec{U}^\dagger\cdot\vec{U} + 2iB_0\vec{U}^\dagger\times\vec{U}\cdot\hat{e}_3T_3, \tag{B-8}$$

where we have used

$$P_{\alpha\beta} = -\frac{1}{4}(1 + \vec{\sigma}_\alpha\cdot\vec{\sigma}_\beta)(1 + \vec{\tau}_\alpha\cdot\vec{\tau}_\beta). \tag{B-9}$$

To evaluate these expressions for incident charged pions, we introduce[49] the pion state vectors

$$\omega^\pm = \mp\frac{1}{\sqrt{2}}(\omega_1 \pm i\omega_2). \tag{B-10}$$

Then, using equation (B-6),

TABLE VIII. List of spin-isospin averages. $\vec{a}$ and $\vec{b}$ are arbitrary vectors. Terms of order 1 have been discarded.

1) $$\sum_{\alpha\neq\beta=1}^{A}(0|\vec{\tau}_\alpha \cdot \vec{\tau}_\beta|0) = -3A$$

2) $$\sum_{\alpha\neq\beta=1}^{A}(0|\vec{\sigma}_\alpha \cdot \vec{\sigma}_\beta|0) = -3A$$

3) $$\sum_{\alpha\neq\beta=1}^{A}(0|\vec{\sigma}_\alpha \cdot \vec{\sigma}_\beta\ \vec{\tau}_\alpha \cdot \vec{\tau}_\beta|0) = -9A$$

4) $$\sum_{\alpha\neq\beta=1}^{A}(0|\tau_{\alpha 3}\tau_{\beta 3}|0) = -A$$

5) $$\sum_{\alpha\neq\beta=1}^{A}(0|\vec{\sigma}_\alpha \cdot \vec{\sigma}_\beta\ \tau_{\alpha 3}\tau_{\beta 3}|0) = -3A$$

6) $$\sum_{\alpha\neq\beta=1}^{A}(0|P_{\alpha\beta}|0) = -\frac{1}{4}A^2 + 4A$$

7) $$\sum_{\alpha\neq\beta=1}^{A}(0|\vec{\tau}_\alpha \cdot \vec{\tau}_\beta\ P_{\alpha\beta}|0) = -\frac{3}{4}A^2$$

8) $$\sum_{\alpha\neq\beta=1}^{A}(0|\vec{\sigma}_\alpha \cdot \vec{\sigma}_\beta\ P_{\alpha\beta}|0) = -\frac{3}{4}A^2$$

TABLE VIII (cont.)

9) $$\sum_{\alpha\neq\beta=1}^{A}(0|\vec{\tau}_\alpha\cdot\vec{\tau}_\beta\,\tau_{\alpha3}\tau_{\beta3}|0) = A^2 + A$$

10) $$\sum_{\alpha\neq\beta=1}^{A}(0|\tau_{\alpha3}\tau_{\beta3}\,P_{\alpha\beta}|0) = -\frac{A^2}{4}$$

11) $$\sum_{\alpha\neq\beta=1}^{A}(0|\vec{\sigma}_\alpha\cdot\vec{\sigma}_\beta\,\tau_{\alpha3}\tau_{\beta3}\,P_{\alpha\beta}|0) = -\frac{3}{4}A^2$$

12) $$\sum_{\alpha\neq\beta=1}^{A}(0|\vec{\sigma}_\alpha\cdot\hat{n}\,P_{\alpha\beta}|0) = 0$$

13) $$\sum_{\alpha\neq\beta=1}^{A}(0|\vec{\sigma}_\alpha\cdot\vec{a}\,\vec{\sigma}_\beta\cdot\vec{b}\,P_{\alpha\beta}|0) = -\frac{1}{4}A^2(\vec{a}\cdot\vec{b})$$

14) $$\sum_{\alpha\neq\beta=1}^{A}(0|\tau_{\alpha3}\,P_{\alpha\beta}|0) = -AT_3$$

15) $$\sum_{\alpha\neq\beta=1}^{A}(0|\vec{\sigma}_\alpha\cdot\hat{n}\,\tau_{\beta3}\,P_{\alpha\beta}|0) = 0$$

16) $$\sum_{\alpha\neq\beta=1}^{A}(0|\vec{\sigma}_\alpha\cdot\vec{a}\,\vec{\sigma}_\beta\cdot\vec{b}\,\tau_{\alpha3}\,P_{\alpha\beta}|0) = -AT_3(\vec{a}\cdot\vec{b})$$

TABLE VIII (cont.)

17) $$\sum_{\alpha\neq\beta=1}^{A}(0|\vec{\sigma}_\alpha \cdot \hat{n} \ \ \vec{\tau}_\alpha \cdot \vec{\tau}_\beta \ P_{\alpha\beta}|0) = 0$$

18) $$\sum_{\alpha\neq\beta=1}^{A}(0|\vec{\sigma}_\alpha \cdot \vec{a} \ \ \vec{\sigma}_\beta \cdot \vec{b} \ \ \vec{\tau}_\alpha \cdot \vec{\tau}_\beta \ P_{\alpha\beta}|0) = -\frac{3}{4} A^2(\vec{a} \cdot \vec{b})$$

$$S = f^2(1 - \frac{1}{A}) - \frac{2}{A} f_\tau^2 \pm \frac{4T_3}{A} ff_\tau,$$

$$S_\tau = - \frac{1}{4} f^2(1 - \frac{16}{A}) - \frac{1}{2} f_\tau^2 \mp \frac{2T_3}{A} ff_\tau,$$

$$\sum_{\alpha=1}^{A} (0|t_\alpha^0|0) = \frac{-A}{(2\pi)^2 \epsilon_0} (f \pm \frac{2T_3}{A} f_\tau),$$

$$\Delta = -2A \left[\frac{f_\tau}{(2\pi)^2 \epsilon_0}\right]^2, \qquad \text{(B-11)}$$

where the upper (lower) sign refers to incident positive (negative) pions, and f and f_τ are defined by equation (4.3).

APPENDIX C

EVALUATION OF THE NUCLEON-NUCLEUS SPIN-ISOSPIN AVERAGES

By charge-independence, the nucleon-nucleon scattering amplitude $f^0_{\alpha c}$ -- in the two-particle CM system -- must have isospin dependence of the form $\vec{I} \cdot \vec{I}$, where

$$\vec{I} = \frac{1}{2} (\vec{\tau}_\alpha + \vec{\tau}_0) \tag{C-1}$$

is the total isospin of the nucleon-nucleon system, and $\vec{\tau}_\alpha$ and $\vec{\tau}_0$ are the isospin operators of the incident and α^{th} nucleon, respectively.

The most general form of $f^0_{\alpha c}$ is then

$$f^0_{\alpha c} = M_\alpha + M'_\alpha \vec{\tau}_\alpha \cdot \vec{\tau}_0 . \tag{C-2}$$

Here, for small scattering angles, M_α and M'_α have spin dependence of the form (see Appendix D):

$$M_\alpha = A_0 + C_0(\vec{\sigma}_0 + \vec{\sigma}_\alpha) \cdot \hat{n} + B_0\vec{\sigma}_0 \cdot \vec{\sigma}_\alpha - H_0\vec{\sigma}_0 \cdot \hat{\ell}\, \vec{\sigma}_\alpha \cdot \hat{\ell} + O(\theta_c^2),$$

$$M'_\alpha = A_\tau + C_\tau(\vec{\sigma}_0 + \vec{\sigma}_\alpha) \cdot \hat{n} + B_\tau\vec{\sigma}_0 \cdot \vec{\sigma}_\alpha - H_\tau\vec{\sigma}_0 \cdot \hat{\ell}\, \vec{\sigma}_\alpha \cdot \hat{\ell} + O(\theta_c^2),$$

(C-3)

and the coefficients $A_0, C_0, \ldots, H_\tau$ are functions of k_0 and $\theta_c = 0$, the momentum and scattering angle, respectively, in the nucleon-nucleon CM system. $\hat{n}$ and $\hat{\ell}$ are unit vectors in the directions $(\vec{k}_0 \times \vec{k})$ and $(\vec{k}_0 + \vec{k})$, respectively.

One can verify the projection operators

$$\Lambda_1 = \frac{1}{4}(3 + \vec{\tau}_\alpha \cdot \vec{\tau}_0),$$

$$\Lambda_0 = \frac{1}{4}(1 - \vec{\tau}_\alpha \cdot \vec{\tau}_0), \tag{C-4}$$

in terms of which equation (C-2) may be written

$$f^0_{\alpha c} = (M_\alpha + M'_\alpha)\Lambda_1 + (M_\alpha - 3M'_\alpha)\Lambda_0 = f^0_{\alpha c}(1)\Lambda_1 + f^0_{\alpha c}(0)\Lambda_0, \tag{C-5}$$

where we have introduced the two independent scattering amplitudes f(1) and f(0). Comparing with equation (5.3), we see that $M_\alpha = f_\alpha$ and $M'_\alpha = f_{\tau\alpha}$. Utilizing equation (A-9), we obtain for $t^0_\alpha t^0_\beta$, evaluated between the state vectors of the incident nucleon,

$$(N^{\pm}|t^{0}_{\alpha}t^{0}_{\beta}|N^{\pm}) = \frac{1}{(2\pi)^4\epsilon_0^2}\left(\frac{q_0}{k_0}\right)^2 \Big\{ M_\alpha M_\beta \pm M_\alpha M'_\beta \,\tau_{\beta 3}$$

$$\pm M'_\alpha M_\beta \tau_{\alpha 3} + M'_\alpha M'_\beta \left[\vec{\tau}_\alpha \cdot \vec{\tau}_\beta \pm i(\vec{\tau}_\alpha \times \vec{\tau}_\beta)_3\right] \Big\}, \qquad \text{(C-6)}$$

where $N^{\pm}$ refers to an incident proton or neutron, respectively.

It is convenient to transform the terms with coefficient C_0, C_τ as follows:

$$C\vec{\sigma} \cdot \hat{n} = C\vec{\sigma} \cdot \frac{\vec{k}_0 \times \vec{k}}{k_0^2 \sin\theta_c} \,. \qquad \text{(C-7)}$$

But it is easily shown that

$$\frac{\vec{k}_0 \times \vec{k}}{k_0^2} = \frac{\vec{q}_0 \times \vec{q}}{q_0^2}\left(\frac{q_0}{k_0}\right). \qquad \text{(C-8)}$$

Therefore, defining

$$C'_0 = \frac{1}{\sin\theta_c} C_0, \qquad C'_\tau = \frac{1}{\sin\theta_c} C_\tau,$$

$$\chi(\vec{q},\vec{q}_0) = \vec{\sigma} \cdot \frac{\vec{q}_0 \times \vec{q}}{q_0^2}\left(\frac{q_0}{k_0}\right), \qquad \text{(C-9)}$$

we write

$$C_0 \vec{\sigma} \cdot \hat{n} = C_0' \, X(\vec{q},\vec{q}_0), \qquad C_\tau \vec{\sigma} \cdot \hat{n} = C_\tau' \, X(\vec{q},\vec{q}_0), \tag{C-10}$$

and C_0' and C_τ' are now of order 1 as $\theta_c \to 0$ (by Appendix D).

Substituting equations (C-10) and (C-3) into (C-6) and using Table VIII to carry out the averages over the nuclear ground state, we find (for unpolarized nuclei)

$$S = \left(\frac{q_0}{k_0}\right)^2 [g_d + h_d \, X(\vec{q},\vec{q}_0)] + O(\theta_c^2),$$

$$S_\tau = \left(\frac{q_0}{k_0}\right)^2 [g_e + h_e \, X(\vec{q},\vec{q}_0)] + O(\theta_c^2),$$

$$\sum_{\alpha=1}^{A} (0|t_\alpha^0|0) = - \frac{A}{(2\pi)^2 \epsilon_0} \left(\frac{q_0}{k_0}\right) \left[(A_0 \pm \frac{2T_3}{A} A_\tau) + (C_0' \pm \frac{2T_3}{A} C_\tau') X(\vec{q},\vec{q}_0) \right],$$

(C-11)

where the spin-independent amplitudes are

$$g_d = (\bar{f})^2 - \frac{1}{A} (\overline{f^2} + 3\overline{f_\tau^2}) \pm \frac{4T_3}{A} \bar{f} \, \bar{f}_\tau,$$

$$g_e = - \frac{1}{4} (\overline{f^2} + 3\overline{f_\tau^2}) + \frac{4}{A} (\bar{f})^2 \mp \frac{2T_3}{A} \overline{ff_\tau} \; . \tag{C-12}$$

Here we have written a superscript bar on the scattering amplitudes [equation (5.3)] to indicate averages over the spin-directions of both particles. That is,

$$\bar{f} = A_0, \qquad \bar{f}_\tau = A_\tau, \qquad (\bar{f})^2 = A_0^2, \qquad (\bar{f}_\tau)^2 = A_\tau^2,$$

$$\overline{f^2} = A_0^2 + 3B_0^2 - 2B_0H_0 + H_0^2$$

$$\overline{f_\tau^2} = A_\tau^2 + 3B_\tau^2 - 2B_\tau H_\tau + H_\tau^2,$$

$$\overline{ff_\tau} = A_0A_\tau + 3B_0B_\tau - B_0H_\tau - B_\tau H_0 + H_0H_\tau. \qquad \text{(C-13)}$$

Finally, the spin-dependent terms are written

$$h_d = A_0C_0' - \frac{1}{A} \left[(A_0 + B_0)C_0' + 3(A_\tau + B_\tau)C_\tau'\right] \pm \frac{2T_3}{A}(A_0C_\tau' + A_\tau C_0'),$$

$$h_e = -\frac{1}{4}\left[(A_0 + B_0)C_0' + 3(A_\tau + B_\tau)C_\tau'\right] + \frac{4}{A}A_0C_0'$$

$$\mp \frac{T_3}{A}\left[(A_0 + B_0)C_\tau' + (A_\tau + B_\tau)C_0'\right]. \qquad \text{(C-14)}$$

All the above expressions are correct through terms of order θ_c and of order A^{-1}. In equations (C-6), (C-11), (C-12), and (C-14) the $(\pm)$ sign refers to incident protons or neutrons, respectively.

Finally,

$$\Delta = -\frac{A}{(2\pi)^4\epsilon_0^2}\left(\frac{q_0}{k_0}\right)^2\left[\overline{f^2} + 3\overline{f_\tau^2} - (\bar{f})^2\right]. \qquad \text{(C-15)}$$

APPENDIX D

EVALUATION OF THE NUCLEON-NUCLEON SCATTERING AMPLITUDES IN TERMS OF PHASE SHIFTS

The nucleon-nucleon scattering amplitude in the CM system may be written as a matrix in spin-space with coefficients which are functions of the scattering angle and the momentum. We use the parametrization of Stapp.[50]

$$M(\vec{k},\vec{k}_0) = A + C(\vec{\sigma}_1 + \vec{\sigma}_2)\cdot\hat{n} + B\vec{\sigma}_1\cdot\hat{n}\vec{\sigma}_2\cdot\hat{n} + \frac{1}{2}G(\vec{\sigma}_1\cdot\hat{m}\vec{\sigma}_2\cdot\hat{m} + \vec{\sigma}_1\cdot\hat{\ell}\vec{\sigma}_2\cdot\hat{\ell})$$
$$+ \frac{1}{2}H(\vec{\sigma}_1\cdot\hat{m}\vec{\sigma}_2\cdot\hat{m} - \vec{\sigma}_1\cdot\hat{\ell}\vec{\sigma}_2\cdot\hat{\ell}), \tag{D-1}$$

where $\hat{n}$, $\hat{\ell}$, and $\hat{m}$ are unit vectors in the directions $(\vec{k}_0 \times \vec{k})$, $(\vec{k}_0 + \vec{k})$, and $(\vec{k} - \vec{k}_0)$, respectively. Stapp[50] has given the general expressions for A, B, C, G, and H in terms of phase shifts for the $I = 1$ state.

For the present applications we are concerned only with small-angle scattering: we need A and C to order θ_c^2; B, G, and H to order θ_c. By using the small-angle expansions for Legendre polynomials,

$$P_\ell(\theta) = 1 - \frac{1}{4}\,\ell(\ell+1)\theta^2 + o(\theta^4),$$

$$P_\ell^{(1)}(\theta) = \frac{1}{2}\,\ell(\ell+1)\ \sin\theta \left[1 - \frac{1}{8}\,(\ell+2)(\ell-1)\theta^2\right] + o(\theta^5),$$

$$P_\ell^{(2)}(\theta) = \frac{1}{8}\,\ell(\ell+2)(\ell+1)(\ell-1)\theta^2 + o(\theta^4), \qquad \text{(D-2)}$$

in Stapp's expressions, the following results are obtained:

$$A(\theta_c) = \frac{1}{4ik_0}\left\{ \sum_{\ell(\text{even})} (2\ell+1)\left[1 - \frac{1}{4}\,\ell(\ell+1)\theta_c^2\right]\alpha_\ell \right.$$

$$\left. + \sum_{\ell(\text{odd})}\left[\sum_{j=\ell-1}^{\ell+1}(2j+1)\alpha_{\ell j}\right]\left[1 - \frac{1}{4}\,\ell(\ell+1)\theta_c^2\right]\right\} + o(\theta_c^4),$$

$$C(\theta_c) = \frac{\sin\theta_c}{4k} \sum_{\ell(\text{odd})}\left[\frac{2\ell+3}{\ell+1}\ \alpha_{\ell,\ell+1} - \frac{2\ell+1}{\ell(\ell+1)}\,\alpha_{\ell\ell}\right.$$

$$\left. - \frac{2\ell-1}{\ell}\,\alpha_{\ell,\ell-1}\right]\ \frac{\ell(\ell+1)}{2}\ \left[1 - \frac{1}{8}\,(\ell+2)(\ell-1)\theta_c^2\right] + o(\theta_c^4)$$

$$= \sin\theta_c\ C'(\theta_c), \qquad \text{(D-3)}$$

$$B(\theta_c) = \frac{1}{4ik}\left\{ \sum_{\ell(\text{odd})}\left[(\ell+1)\alpha_{\ell,\ell+1} + \ell\alpha_{\ell,\ell-1} + \sqrt{(\ell+1)(\ell+2)}\ \alpha^{\ell+1}\right.\right.$$

$$\left.\left. + \sqrt{\ell(\ell-1)}\ \alpha^{\ell-1}\right] - \sum_{\ell(\text{even})} (2\ell+1)\alpha_\ell \right\} + o(\theta_c^2),$$

$$G(\theta_c) = \frac{1}{4ik}\left\{\sum_{\ell(\mathrm{odd})}\left[(\ell+2)\alpha_{\ell,\ell+1} + (2\ell+1)\alpha_{\ell\ell} + (\ell-1)\alpha_{\ell,\ell-1}\right.\right.$$

$$\left.\left. - \sqrt{(\ell+1)(\ell+2)}\,\alpha^{\ell+1} - \sqrt{\ell(\ell-1)}\,\alpha^{\ell-1}\right] - 2\sum_{\ell(\mathrm{even})}(2\ell+1)\alpha_{\ell}\right\} + O(\theta_c^2),$$

$$H(\theta_c) = \frac{1}{4ik}\sum_{\ell(\mathrm{odd})}\left[\ell\,\alpha_{\ell,\ell+1} - (2\ell+1)\alpha_{\ell\ell} + (\ell+1)\alpha_{\ell,\ell-1}\right.$$

$$\left. + 3\sqrt{(\ell+1)(\ell+2)}\,\alpha^{\ell+1} + 3\sqrt{\ell(\ell-1)}\,\alpha^{\ell-1}\right] + O(\theta_c^2).$$

(D-4)

The $\alpha_{\ell j}$ are most conveniently expressed in terms of the "bar" phase shifts of Stapp[36,50] as follows:[13] for the singlet state

$$\alpha_{\ell} = e^{2i\delta_{\ell}} - e^{2i\varphi_{\ell}}$$

and similarly for the triplet state with $\ell = j$, while for the other triplet states

$$\alpha_{\ell j} = e^{2i\bar{\delta}_{\ell j}}\cos 2\bar{\epsilon}_j - e^{2i\varphi_{\ell}} \qquad (j = \ell\pm1),$$

$$\alpha^{j} = i\sin 2\bar{\epsilon}_j\left[e^{i(\bar{\delta}_{j+1,j} + \bar{\delta}_{j-1,j})}\right],$$

and φ_{ℓ} is the coulomb phase shift.

Comparing the above equations (D-4), we see

$$G + H = 2B + O(\theta_c^2). \tag{D-5}$$

Using this in equation (D-1), we obtain the expressions given in equations (C-3).

The coulomb scattering effects have been neglected here, as the optical potential proper deals only with the purely nuclear part of the scattering. The effects of coulomb scattering are then to be considered when the scattering from the optical potential is calculated, as described by several authors.[51]

The coefficients of the I = 0 scattering amplitude are also obtained from equations (D-3) and (D-4) by interchanging "even" and "odd."

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